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Stable Hybrid Model Predictive Control for Ramp Metering

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***Abstract*— We formulate the Asymmetric Cell Transmission Model (ACTM) as a piecewise affine system defined over regions of the state and input space. We synthesize a hybrid Model Predictive Controller (MPC) for the piecewise affine model such that persistent feasibility and stability are guaranteed. We do so by designing a terminal constraint and terminal cost for the equilibrium state of the system. We include a detailed analysis of the equilibrium point of the piecewise affine system, and define the region of demand under which an equilibrium point exists. We show that our method achieves the same performance in terms of efficiency and exhibits much smoother behavior than that of the commonly used relaxed ACTM controller formulation.**

I. INTRODUCTION

Due to the inflation of congestion and delays in urban vehicular networks [1], development of traffic management and control tools is becoming more and more crucial for efficient utilization of the current infrastructure of metropolitan areas. There are two types of networks: freeways and urban arterials, with each type having its own model of traffic progression. In this work, we study the synthesis of control strategies that could be incorporated for ameliorating traffic conditions in freeways. In freeways, the inputs available to the control engineer include ramp metering and variable speed limits. Ramp metering refers to the determination of the amount of flow entering the freeway through on-ramps; variable speed limits are a mechanism of assigning speed limits dynamically such that congestion occurrence is postponed or avoided.

Ramp metering has been shown to be capable of reducing delays in freeways [2]; hence it is an effective strategy for increasing throughput of the network. As a result of this effectiveness, several ramp metering strategies have been developed ranging from fixed–time controllers [2] to controller synthesis from signal temporal logic specifications [3] to synthesis of robust state-feedback controllers [4]. Among possible approaches for ramp metering, model–predictive–based controllers that optimize some performance metric are favored. In [5], a nonlinear model predictive controller minimizing a weighted sum of Total Travel Time (TTT) and Total Time Spent (TTS) was introduced. In [6], variable speed limits are taken into account in a nonlinear model predictive controller with a TTS cost function. In [7], ramp metering flows are assumed to be obtained through an existing strategy, and the speed limits are attained through a model predictive controller. In [8], a weighted sum of TTT and Total Travel Distance (TTD) is optimized at every time step, where the optimization problem of interest is formulated as a linear program. In [9], speed limits are considered in addition to ramp metering while optimizing for the same performance measure as [8].

Despite the popularity and practicality of model predictive control, stability and feasibility properties are often not addressed in practice. The existing work in the traffic network control literature for handling infeasibility conditions include [10] which proposes to minimize the amount of constraint violation of temporal logic specifications for signalized intersections and [11], where persistent feasibility for a target set of states is guaranteed by constructing invariant sets of a finite abstraction of the system (an approximation of the system evolution). Both [10] and [11] are for urban arterial control; consequently, for freeways, the issues of persistent feasibility and stability under model predictive control are still not addressed. Hence, the focus of this work is to design model predictive controllers for traffic networks with guarantees on persistent feasibility and closed–loop stability of the controller.

The organization of this paper is as follows. First, we introduce the Asymmetric Cell Transmission Model (ACTM) used to describe the freeway system and describe it as a piecewise affine system over regions of the state and input. Next, we show how to develop a hybrid model predictive controller for this system by computing an equilibrium point and designing a terminal set and terminal cost. Then, we compare our proposed strategy to the popular relaxed linear programming approach. The paper concludes with some ideas subject to future work.

II. ASYMMETRIC CELL TRANSMISSION MODEL

In this section, we describe the Asymmetric Cell Transmission Model (ACTM) [8] which is adopted throughout this paper for modeling freeway traffic. For ACTM, the freeway is divided into I segments where each segment can have at most one onramp and one offramp as depicted in Figure 1. The index set of all the segments is denoted as $\mathcal{I} \triangleq \{1, \ldots, I\}$.

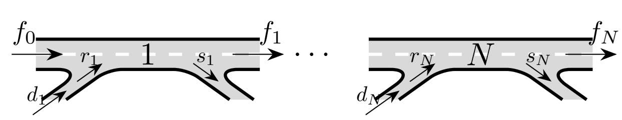

Fig. 1. Schematic of Freeway Segmentation

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For each segment i, the following quantities are defined:

- $n_i(k)$: Number of vehicles in segment i at time step k. This is also referred to as the density of vehicles in segment i.
- $f_i(k)$: Number of vehicles leaving segment i, moving to segment $i+1$ during time step k.
- $l_i(k)$: Number of vehicles queuing on the onramp of segment i at time step k.
- $r_i(k)$: Number of vehicles entering segment i through its onramp during time step k.
- $d_i(k)$: Number of vehicles entering onramp of segment i during time step k, also known as demand.

In this model, mainline and onramp densities ($n_i(k)$ and $l_i(k)$, $i \in \mathcal{I}$) are the states of the system. The following parameters of the model are assumed to be known a priori through a calibration process [12]. The parameters are positive, unless otherwise denoted.

- v_i: Normalized free-flow speed $\in (0, 1]$.
- β_i: Split ratio of the offramp in segment $i \in [0, 1]$.
- w_i: Normalized congestion wave speed $\in (0, 1]$.
- γ: Blending coefficient of onramp flows $\in [0, 1]$.
- n_i^c: Critical density of segment i, the density above which segment i is considered congested.
- $f_0(k)$: The exogenous demand entering link 1 (the most upstream segment) at time step k.

The dynamics of the system states are obtained through mass conservation equations:

$$n_i(k+1) = n_i(k) + f_{i-1}(k) + r_i(k) - f_i(k)/\bar{\beta}_i, \quad (1)$$

$$l_i(k+1) = l_i(k) + d_i(k) - r_i(k). \quad (2)$$

where $\bar{\beta}_i = 1 - \beta_i$. Note that the onramp flows $r_i(k)$ are determined by a controller (assuming that all onramps are actuated). The last piece of the traffic evolution model is the mapping function between mainline flows $f_i(k)$ and densities $n_i(k)$.

In freeway first order models, the flow in each segment i is restricted by the number of vehicles available to leave segment i, ($\bar{\beta}_i v_i \left(n_i(k) + \gamma r_i(k)\right)$), mainline capacity ($\bar{f}_i$) and the available space in the downstream segment $i+1$, ($w_{i+1}\left(\bar{n}_{i+1} - n_{i+1}(k) - \gamma r_{i+1}(k)\right)$). In other words, the flow in segment $i \in \{1, \ldots, I-1\}$ is computed by:

$$\begin{aligned} f_i(k) = \min\{&\bar{\beta}_i v_i \left(n_i(k) + \gamma r_i(k)\right), \\ &w_{i+1}\left(\bar{n}_{i+1} - n_{i+1}(k) - \gamma r_{i+1}(k)\right), \bar{f}_i\}. \end{aligned} \quad (3)$$

Note that the edge flows of the freeway system are special cases. The incoming flow $f_0(k)$ is an exogenous input. The outgoing flow $f_I(k)$ is simplified since w_{I+1} is undefined:

$$f_I(k) = \min\{\bar{\beta}_i v_i \left(n_i(k) + \gamma r_i(k)\right), \bar{f}_i\}. \quad (4)$$

Without loss of generality, we assume that:

$$\bar{f}_i = \bar{\beta}_i v_i n_i^c = w_{i+1}(\bar{n}_{i+1} - n_{i+1}^c) \quad (5)$$

where n_i^c is the critical density of segment i which is the density at which segment i transitions between congested and uncongested states. This is a common assumption in freeway modeling [13].

The densities $n_i(k)$, queue lengths $l_i(k)$, flows $f_i(k)$, and onramp inputs $r_i(k)$ are bounded by box constraints. The states $n_i(k)$, $l_i(k)$, $f_i(k)$ are non-negative since a negative density, queue length, or flow between segments are not physically possible. Similarly, the onramp inputs $r_i(k)$ are restricted to be non-negative. In addition, each state and input has an upper bound, defined here:

- $\bar{n}_i$: Jam density of segment i, the maximum number of vehicles that can fit in segment i.
- $\bar{l}_i$: Max queue length of segment i.
- $\bar{f}_i$: Main-line capacity defined as the maximum number of vehicles that can leave segment i.
- $\bar{r}_i$: Maximum allowable ramp flows.

The density and queue lengths are restricted simply by space limitations on the freeway and onramp, respectively. The flows f_i and the onramp inputs r_i are restricted by the capacity of the freeway; only a certain amount of cars can move forward on the given freeway lanes in the sampling time of the model.

A. Piecewise Affine Model

The minimum operator in Equation (3) implies that freeway can be described by a piecewise affine model: in free flow, $f_i(k) = \bar{\beta}_i v_i \left(n_i(k) + \gamma r_i(k)\right)$; in congestion, $f_i(k) = w_{i+1}\left(\bar{n}_{i+1} - n_{i+1}(k) - \gamma r_{i+1}(k)\right)$. This piecewise affine system is described by the Equations (1), (2) and (3). Next, we specify the state space partitions affiliated with each piece of the system. The dynamics of segment i are dependent on segment i's current density $n_i(k)$ and the density of the segment directly downstream $n_{i+1}(k)$. Define the uncongested flow $f_{v,i}(k)$ of segment i and the congested flow $f_{w,i}(k)$:

$$f_{v,i}(k) = \bar{\beta}_i v_i (n_i(k) + \gamma r_i(k)) \quad (6)$$

$$f_{w,i}(k) = \begin{cases} w_{i+1}(\bar{n}_{i+1} - n_{i+1}(k) - \gamma r_{i+1}(k)) & \text{if } i < I \\ \bar{f}_i & \text{if } i = I \end{cases} \quad (7)$$

The last segment, when $i = I$, is different from other segments as there is no downstream segment imposing restrictions to its flow. Hence, the flow $f_i(k)$ can be characterized by three regions:

$$f_i(k) = \begin{cases} f_{v,i}(k) & \text{if } f_{v,i(k)} \le f_{w,i}(k), f_{v,i(k)} \le \bar{f}_i \\ f_{w,i}(k) & \text{if } f_{w,i(k)} \le f_{v,i}(k), f_{w,i(k)} \le \bar{f}_i \\ \bar{f}_i & \text{if } \bar{f}_i \le f_{v,i(k)}, \bar{f}_i \le f_{w,i(k)}. \end{cases} \quad (8)$$

In general, the regions for flow $f_i(k)$ defined in (8) depend on the state $n_i(k)$ and $n_{i+1}(k)$ (if $i < I$) as well as the input $r_i(k)$. If $\gamma = 0$, then the regions defining $f_i(k)$ are not dependent on the input $r_i(k)$.

The dynamics of $n_i(k+1)$ rely on both $f_{i-1}(k)$ and $f_i(k)$ resulting in nine equations for the density of segment i. The exception to this is for the first segment $i = 1$ and the last segment $i = I$. The first segment includes $f_0(k)$ as an external demand, so there are only three equations for $n_1(k+1)$. The density of the last segment is simplified to six equations since $f_I(k)$ is only defined by two regions (4).

These equations hold for a network of any size. The flow variables $f_i(k)$ can be eliminated by substituting the Eq. (8) into (1) to achieve a piecewise controller over state space regions $\mathcal{R}_i$. The dynamics are then defined by equation (2) and

$$N(k+1) = A_i N(k) + B_i R(k) + g_i \text{ if } (N(k), R(k)) \in \mathcal{R}_i \quad (9)$$

where $N(k) = [n_1(k), n_2(k), \dots, n_I(k)]^T$ and $R(k) = [r_1(k), r_2(k), \dots, r_I(k)]^T$. Denoting the vector $L(k) = [l_1(k), \dots, l_I(k)]^T$, $D(k) = [d_1(k), \dots, d_I(k)]^T$ the vector form of equation (2) is

$$L(k+1) = L(k) + D(k) - R(k). \tag{10}$$

Note that in this notation, we have assumed that all segments have onramps, and all onramps are actuated. If this is not the case, the equations can easily be modified so that the ramp queues of actual onramps appear in the equations.

III. Hybrid Model Predictive Control

In this section, we construct a hybrid model predictive controller for the ramp metering system. Our approach is to design a model predictive controller whose objective is to track the equilibrium point. Then, we can apply hybrid model predictive control design techniques to synthesize the terminal set and terminal cost to guarantee persistent feasibility and stability of the piecewise affine system.

For hybrid model predictive control, a constrained finite time optimal control problem is solved at each time step. The first input is applied, then the horizon is receded and the shifted horizon problem is then solved at the next time step using the current state feedback. The H-step constrained finite time optimal control problem at time step t is:

$$\min_{\tilde{X}(k),\tilde{U}(k)} \quad \tilde{X}(H)^T P \tilde{X}(H) + \sum_{k=0}^{H-1} \tilde{X}(k)^T Q \tilde{X}(k) + \tilde{U}(k)^T R \tilde{U}(k) \tag{11a}$$

$$\begin{aligned}
s.t.\quad & (9), (10) && (11b)\\
& 0 \le N(k) \le \bar{N} && (11c)\\
& 0 \le L(k) \le \bar{L} && (11d)\\
& 0 \le R(k) \le \bar{R} && (11e)\\
& \tilde{X}(H) \in \mathcal{X}_f && (11f)\\
& N(0) = N(t), L(0) = L(t) && (11g)
\end{aligned}$$

where $\tilde{X}(k) = [N(k)^T - N_{ss}^T, L(k)^T - L_{ss}^T]^T$, $\tilde{U}(k) = R(k) - R_{ss}$, $\bar{N} = [\bar{n}_1, \dots, \bar{n}_I]^T$, $\bar{L} = [\bar{l}_1, \dots, \bar{l}_I]^T$, $\bar{R} = [\bar{r}_1, \dots, \bar{r}_I]^T$. The steady state or equilibrium points are denoted as N_{ss}, L_{ss}, R_{ss}. The cost is quadratic with state penalty $Q = Q^T \succ 0$, input penalty $R = R^T \succ 0$, and terminal cost penalty $P = P^T \succ 0$. The final state is restricted to be in the terminal set $\mathcal{X}_f$. The current state feedback at time t, $N(t)$ and $L(t)$ are used. The MPC control law for ramp metering at time t is then:

$$R(t) = \tilde{U}^*(0) + R_{ss} \tag{12}$$

where $\tilde{U}^*(0)$ is the optimal input for the first step computed by (11).

We next show how to compute the equilibrium of the system for a constant demand profile. Then, we design a stable and persistently feasible Hybrid MPC by applying the design methodology proposed in [14]-[15] to our problem.

A. Equilibrium Computation

The equilibrium of the piecewise system (9) is dependent on the current demand. In order to evaluate an equilibrium of the system, we assume the demand is stationary or constant over time, i.e. $d_i(k) = d_i \; \forall k$ and $f_0(k) = f_0 \; \forall k$. Denote the constant demand vector $D = [d_1, \dots, d_I]^T$.

First, without knowing which region $\mathcal{R}_i$ of the model (9) the equilibrium point lies in, the steady state queue length L_{ss} and ramp metering input R_{ss} can be determined by computing the steady state of (10):

$$L_{ss} = L_{ss} + D - R_{ss}. \tag{13}$$

This equations tells us that the steady state ramp metering input $R_{ss} = D$ and that L_{ss} is a constant value.

Remark 1. *A steady state ramp metering input $R_{ss} = D$ is feasible if $0 \le D \le \bar{R}$ where $\bar{R}$ is the vector of upper bound limits for R, i.e. $\bar{R} = [\bar{r}_1, \dots, \bar{r}_I]^T$.*

It can be verified by examining equation (10) that if D exceeds the feasible values for R, L does not have an equilibrium point. That is, if $d_i > \bar{r}_i$, then queue i is receiving more cars than it can release onto the freeway. If the demand is stationary, then the queue length l_i will grow without bound and is therefore unstable.

Remark 2. *A steady state queue length vector L_{ss} is feasible if $0 \le L_{ss} \le \bar{L}$ where $\bar{L}$ is the vector of upper bound limits for L, i.e. $\bar{L} = [\bar{l}_1, \dots, \bar{l}_I]^T$. The steady state queue length for segment i, $[l_i]_{ss} = l_i(0)$ if $d_i = \bar{r}_i$. Otherwise, the steady states $[l_i]_{ss} \in [0, \bar{l}_i]$ are reachable.*

Remark 3. *Among a continuum of equilibria for L, the desired equilibrium point is $L_{ss} = 0$. This corresponds to empty queues at steady state, which guarantees that no cars are getting stuck in a queue (cars are stuck if $L > 0$ and $R = 0$ for a long period of time). However, in order to prove stability of the hybrid MPC (see the details in the next section), the equilibrium point must be interior to the constraint set. Therefore, we choose an equilibrium point of $L_{SS} = \epsilon$, where ϵ is a positive number close to zero. In practice, a controller that brings the queue length to ϵ effectively brings L to zero.*

Next, we review some results from the literature regarding the steady state density vector N_{ss}. In [13], the equilibria of the Cell Transmission Model (CTM) under stationary demand is studied. The CTM is similar to ACTM except that $\gamma = 0$ and there are no ramp queues $L(k)$. Equilibrium results are established under the assumption of *feasible demand*. We provide a definition of a *feasible demand* that is similar to that from [13] in Definition 1. Under the assumption of *feasible demand*, the work in [13] proves that there exists a unique *uncongested* equilibrium that is stable in the sense of Lyapunov. An *uncongested* equilibrium is an equilibrium point in which N_{ss} is in the set $[0, N^c]$ where $N^c = [n_1^c, \dots, n_I^c]^T$.

Definition 1 (cf. [13])**.** *A constant demand $D = [d_1, \dots, d_I]^T$ is **feasible** if there exists an equilibrium point of the system defined by* (2), (1), (3)*, the resulting flows are feasible, i.e. $[f_i]_{ss} \in [0, \bar{f}_i] \; \forall i \in \mathcal{I}$, and the ramp metering inputs are nonnegative, i.e. $[r_i]_{ss} \ge 0 \; \forall i \in \mathcal{I}$.*

Lemma 1. *For a stationary demand D that is feasible in the sense of Definition 1, there exists an uncongested equilibrium of the ACTM model.*

Proof. The proof follows directly from [13] Lemma 4.1. □

Lemma 1 extends Lemma 4.1 of [13] to the more general ACTM model (γ is not restricted to be zero). Let $\mathcal{R}_1 \equiv \{n_i(k), r_i(k) | f_{v,i}(k) \le f_{w,i}, f_{v,i}(k) \le \bar{f}_i \ \forall i \in \mathcal{I}\}$. We next show that the uncongested equilibrium N_{ss} is in $\mathcal{R}_1$.

Lemma 2. *If there is an uncongested equilibrium point N_{ss}, then it is in $\mathcal{R}_1$.*

Proof. Following the proof from [13] (Lemma 4.1 proof), the uncongested equilibrium is:

$$n_i = (\bar{\beta}_i v_i)^{-1} f_i - \gamma r_i. \tag{14}$$

Notice that:

$$\begin{aligned} n_i &\le (\bar{\beta}_i v_i)^{-1} \bar{f}_i - \gamma r_i \\ &\le n_i^c - \gamma r_i. \end{aligned}$$

Then,

$$\begin{aligned} f_{v,i} &= \bar{\beta}_i v_i (n_i + \gamma r_i) \\ &\le \bar{\beta}_i v_i (n_i^c - \gamma r_i + \gamma r_i) \\ &\le \bar{\beta}_i v_i n_i^c = \bar{f}_i. \end{aligned}$$

Recall also that $\bar{\beta}_i v_i n_i^c = w_{i+1}(\bar{n}_{i+1} - n_{i+1}^c)$ from (5). Then,

$$\begin{aligned} f_{v,i} &\le \bar{\beta}_i v_i n_i^c \\ &= w_{i+1}(\bar{n}_{i+1} - n_{i+1}^c) \\ &\le w_{i+1}(\bar{n}_{i+1} - n_i - \gamma r_i) \\ &= f_{w,i}. \end{aligned}$$

The second-to-last line is true because $n_i \le n_i^c - \gamma r_i \ \forall i \in \mathcal{I}$, so $n_{i+1}^c \ge n_{i+1} + \gamma r_{i+1}$. Thus, $f_{v,i} \le f_{w,i}, f_{v,i} \le \bar{f}_i \ \forall i \in \mathcal{I}$. This is the definition of $\mathcal{R}_1$, so the uncongested equilibrium must lie in $\mathcal{R}_1$. □

The piecewise affine model defined in Section II-A enables the development of a set definition assumption about the demand. The steady state density vector N_{ss} can be found by computing the equilibrium of the affine system defined over $\mathcal{R}_1$, or by solving the following equation:

$$N_{ss} = A_1 N_{ss} + B_1 R_{ss} + g_1. \tag{15}$$

In order for the steady state equation 15 to hold, the resulting solution N_{ss}, R_{ss} must lie in the uncongested region $\mathcal{R}_1$. This brings us to the following theorem.

Theorem 1. *Assume the following about the demand D:*

- $0 \le D \le \bar{R}$
- *The initial state $L(0)$ is feasible*
- *Demand D and and external flow f_0 satisfy $\big((I - A_1)^{-1}(B_1 D + g_1), D\big) \in \mathcal{R}_1$.*

Then, there exists a feasible equilibrium N_{ss}, R_{ss}, L_{ss} of the system (9)-(10) *such that $N_{ss} \in [0, N^c]$, $R_{ss} \in [0, \bar{R}]$, $L_{ss} \in [0, \bar{L}]$. Moreover, there exists an equilibrium that is uncongested, i.e. an equilibrium lies in $\mathcal{R}_1$.*

Proof. The first two conditions give us feasible equilibrium vectors R_{ss} and L_{ss}, following Remarks 1 and 2. The last condition gives a steady state N_{ss} that lies in $\mathcal{R}_1$. Note that the matrix $(I - A_1)$ is invertible with eigenvalues $\{v_i\}$. □

Theorem 1 differs from the previous work [13] in that it precisely defines the set of demands that result in having an equilibrium point in the uncongested region, $\mathcal{R}_1$. The particularly powerful result of Lemmas 1 and 2 is that a search through other regions $\mathcal{R}_i$ with $i \ne 1$ is not necessary. This is because an uncongested equilibrium must lie in $\mathcal{R}_1$.

Lemma 3. *If the assumptions of Theorem 1 hold, and the demands D and f_0 are such that $0 < D < \bar{R}$ and $(I - A_1)^{-1}(B_1 D + g_1) > 0$, then, the equilibrium N_{ss}, R_{ss}, L_{ss} of the system* (9)-(10) *is interior to the constraint set, i.e. $N_{ss} \in (0, \bar{N})$, $R_{ss} \in (0, \bar{R})$, $L_{ss} \in (0, \bar{L})$.*

Proof. The equilibrium $R_{ss} \in (0, \bar{R})$ by Remark 1. The equilibrium $L_{ss} \in (0, \bar{L})$ by Remark 2. The equilibrium $N_{ss} > 0$ since $N_{ss} = (I - A_1)^{-}1(B_1 D + g_1) > 0$. The equilibrium $N_{ss} < \bar{N}$ since by Lemma 1, the equilibrium point $N_{ss} \le N^c$ and by definition $N^c < \bar{N}$. □

B. Design of Terminal Set and Terminal Cost

It is well known that careful design of the terminal cost and terminal constraint of a receding horizon controller yields persistent feasibility and stability results (see Theorem 13.2 of [16]). In this section, We develop a stable and persistently feasible hybrid model predictive controller based on the methods of [14] and [15]. In particular, we follow the design of Algorithm 3.1 in [14].

The design of a model predictive controller requires that the equilibrium point must lie in the interior of the constrained state space, i.e. $0 < N_{ss} < \bar{N}$, $0 < L_{ss} < \bar{L}$, $0 < R_{ss} < \bar{R}$ (see Assumption (A1) in Theorem 13.2 of [16]; also see [15]). We make the assumptions about demand in Lemma 3.

First, we identify the regions of the dynamics in which the equilibrium lies. In many cases, the equilibrium lies in the interior of region $\mathcal{R}_1$, but it may generally lie on the border of multiple regions. Define the error dynamics of the regions $\mathcal{S}_{eq}$ which contain the equilibrium point:

$$\tilde{X}(k+1) = \tilde{A}_i \tilde{X}(k) + \tilde{B}_i \tilde{U}(k) \qquad i \in \mathcal{R}_i, \mathcal{R}_i \subseteq \mathcal{S}_{eq}. \tag{16}$$

Note that the error dynamics are in general a piecewise linear system.

Then, a search is done to find stabilizing piecewise linear feedback controllers $\tilde{U}(k) = K_i \tilde{X}(k)$ for all regions in $\mathcal{S}_{eq}$ and an associated Lyapunov function $V(\tilde{X}) = \tilde{X}^T P \tilde{X}$. This search can be done by solving the following semidefinite program:

$$\min_{Z, Y_i, \gamma} \ \gamma \tag{17a}$$

$$\text{s.t. } Z \succ 0 \tag{17b}$$

$$\begin{bmatrix} Z & (\tilde{A}_i Z + \tilde{B}_i Y_i) & Q^{1/2} Z & R^{1/2} Z \\ (\tilde{A}_i Z + \tilde{B}_i Y_i)^T & Z & 0 & 0 \\ Q^{1/2} Z & 0 & \gamma I & 0 \\ R^{1/2} Z & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\forall i \in \mathcal{R}_i \subseteq \mathcal{S}_{eq} \tag{17c}$$

where $Y_i = K_i Z$, $Z = P^{-1}/\gamma$

If the equilibrium lies in the interior of $\mathcal{R}_1$, then the terminal cost P can be designed by solving the Algebraic Riccati Equation for the $\tilde{A}_1$ and $\tilde{B}_1$ matrices defined in $\mathcal{R}_1$ (cf. Remark 2 in [14]). Moreover, the terminal set can be found by computing the maximal positive invariant set for the system in $\mathcal{R}_1$ subject to the LQR controller found for $\tilde{A}_1, \tilde{B}_1, Q, R$.

In general, the maximal positive invariant set for the system under the controllers K_i found by (17) is used as the terminal set $\mathcal{X}_f$. In general, the maximal positive invariant set is intensive to compute as it is comprised of a union of polyhedron; see [15] for how to compute the maximal positive invariant set. We use tools from the multi-parametric toolbox [17] to compute the maximal positive invariant set.

By Theorem 3.1 of [14], the closed loop system is asymptotically stable under the hybrid model predictive controller (11)-(12) with the terminal cost P determined by (17) and the terminal set the maximal positive invariant set of this system under piecewise linear feedback controllers K_i.

C. Practical Considerations

We have designed a persistently feasible and stable controller. To the best of our knowledge, no other proposed controller for this problem has any feasibility or stability guarantees. There are however, a few practical considerations that must be taken into account.

The first consideration is how to determine the prediction horizon N such that the set of feasible initial states covers the full state space. The initial feasible state can be computed for an N step horizon by computing the controllable pre-sets of the piecewise affine system (9), (10). The desired prediction horizon N would correspond to the N-step controllable pre-set that covers the full state space. However, these pre-sets are in general a union of polyhedrons and thus are hard to compute [15] since a union of polyhedrons is not necessarily convex and the union may consist of many polyhedrons. It may be easier in some cases to determine N in simulation by checking how big N must be to be feasible for the worst case initial feasible states.

The second consideration is that the assumptions about D in Theorem (1) may be too restrictive. In reality, it is possible to allow the queues to spillover into the urban arterial networks. This effectively allows the upper bound constraint on L to be a soft constraint. Simply adding a soft constraint extends the feasibility of the problem, but loses any guarantees about stability. In general, stability of soft constrained MPC is understood for linear systems [18], [19]. Unfortunately, designing stable MPC controllers for soft constrained piecewise affine systems has not yet been studied to the best of our knowledge.

IV. Case Studies

In this section, we present simulation results comparing our hybrid MPC approach with the linear program (LP) formulation of optimal freeway ramp metering [8] for a two link freeway example with a sampling time of 3 seconds. The prediction horizon for both controllers is $H = 25$. Figure 2 demonstrates the trajectory of states and inputs when our hybrid MPC framework is utilized for ramp metering whereas Figure 3 shows the same plots where the linear programming formulation is used.

We evaluate the efficiency of the two controllers using two metrics: total travel time (TTT) and total travel distance

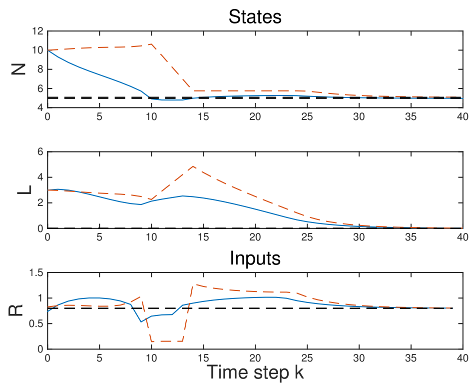

Fig. 2. Trajectory of Hybrid MPC controller. The blue solid and red dashed lines represent trajectories of the first and second segments respectively.

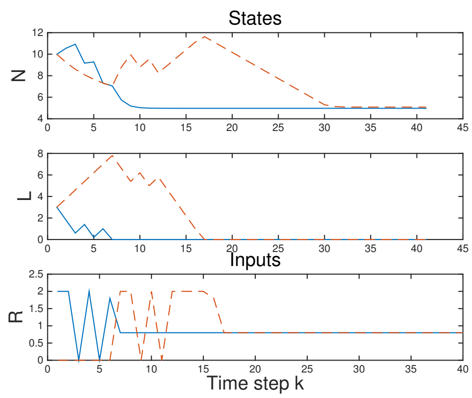

Fig. 3. Trajectory of LP controller

(TTD). The total travel time can be written as the sum of the segment densities and the queue lengths over all time:

$$TTT = \sum_{i=1}^{I} \sum_{k=0}^{\infty} \big(n_i(k) + l_i(k)\big), \tag{18}$$

and the total travel distance (which is a measurement of throughput) is equal to the sum of the segment flows and on-ramp flows over all time:

$$TTD = \sum_{i=1}^{I} \sum_{k=0}^{\infty} \big(f_i(k) + r_i(k)\big). \tag{19}$$

In general, a small total travel time and a large total travel distance are desired.

As seen in figures 2 and 3, the initial condition is in congestion. Both controllers can successfully steer the freeway to the uncongested regime and an uncongested equilibrium is achieved. The total travel time (TTT) of the hybrid trajectory over the simulation period is 648.3 vehicles while the total travel time of the trajectory resulting from linear program formulation is 638.3 vehicles. Thus, the linear program formulation is 1.6% more efficient in terms of total travel time. However, the total travel distance (TTD) is the same for both controllers at 353.1 vehicles.

The trajectories of the two controllers are quite different qualitatively. Figures 2 and 3 show that the hybrid MPC trajectory is much smoother than the LP trajectory, and there are less variations of the control input for the hybrid controller. This smoothness is advantageous in a real-time implementation since rapidly changing ramp flows cannot be achieved in practice; this is because traffic response time is longer than the agility required by linear program formulation.

We note that our proposed controller has guarantees by design whereas the linear programming method does not. Our controller has a terminal set and terminal cost design which guarantee the stability and persistent feasibility of the controller. For the linear program, a cool–down period and long prediction horizon are used to encourage stability [8]. There are no guarantees of persistent feasibility attached to the linear programming controller. Moreover, our controller is guaranteed to reach the equilibrium point whereas the convergence of the linear program must be checked for various prediction horizons and various initial conditions.

Another issue with the linear programming approach is the exactness of the model relaxation. Even if the horizon length and cool down period length are properly designed, there are scenarios where the relaxation of the ACTM model does not match the nonlinear model exactly. For example, if the ramp flows R are not strictly less than the available mainline space allocated to onramps, the relaxations introduced in [8] do not necessarily hold. This results in model mismatch between the predicted model of the controller and the evolution of the actual network. Hence, in such cases, the linear program does not perform optimally, while the hybrid formulation is not restricted to certain conditions on the mainline densities.

V. Conclusions and Future Work

In conclusion, we have designed the first stable model predictive controller for the ramp metering problem. To the best of our knowledge, no other ramp metering model predictive controller has rigorous guarantees of stability and persistent feasibility. We compare our controller to the widely proposed linear programming solution and show that there is no significant performance degradation of our design. We show that our controller provides a smooth trajectory and converges quite quickly to the equilibrium point.

We acknowledge that in general the demand may vary throughout the course of a day. However, real data shows that the demand is constant for long periods of the day. In ramp metering research, the demand is often assumed to be constant over a prediction horizon. Nevertheless, the handling of time varying demand is of interest. It is common practice in the model predictive control literature to utilize an observer and constant demand model. This design is able to achieve desired steady state convergence (see Theorem 13.4 in [16]).

Finally, the scalability of a hybrid model predictive control approach is subject to future work. However, the control design done in this work makes large-scale methodology such as decentralized, hierarchical, and distributed model predictive control more easily applicable. In particular, it is now possible to design and analyze stability of large-scale algorithms for the ramp metering problem.

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