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# Stability and Robustness Analysis of a Class of Adaptive Controllers for Robotic Manipulators

## Abstract

*The stability and robustness properties of the adaptive control scheme proposed by Sadegh and Horowitz (1987) are studied. The properties include the global exponential stability and $\mathbf{L}_p$ input/output stability of the nonadaptive (i.e., fixed-parameter) control system and the global asymptotic stability of the adaptive control scheme. Sufficient conditions for the convergence of the estimated parameters to their true values are also given. A computationally efficient adaptation scheme that is a modified version of the original scheme is proposed. The modified scheme utilizes the desired trajectory outputs, which can be calculated a priori, instead of the actual joint outputs in the parameter adaptation algorithm and the nonlinearity compensation controller. Sufficient conditions for guaranteeing all the stability properties of the original scheme in the modified scheme are also explicitly derived. A computer simulation study of the performance of both schemes in the presence of noise disturbances is conducted.*

## 1. Introduction

Robotic manipulators are mechanical systems with inherently nonlinear dynamic characteristics. Furthermore, their inertia properties and gravitational loads vary during operation and are dependent on the manipulator payload, which may not be necessarily known in advance. In flexible automation environments, robotic manipulators are required to handle a variety of tasks while maintaining a consistently high performance. Adaptive control has been proposed as a viable technique for achieving a consistent manipulator dynamic behavior in the presence of configuration and payload variations.

The earliest work on model reference adaptive control for manipulators (Dubowsky and DesForges 1979) is based on a linear decoupled model. The steepest descent technique is utilized for parameter adaptation. The first work on adaptive controls of mechanical manipulators based on stability theories was reported by Horowitz and Tomizuka (1980). In this work the hyperstability theory was utilized to derive an adaptive control law for linearizing and decoupling the nonlinear manipulator dynamics. Experimental evaluations of this approach have been reported by Anex and Hubbard (1984), Tomizuka et al. (1986), and Horowitz et al. (1987). A Lyapunov stability-based adaptive control approach for trajectory following has been proposed by Takegaki and Arimoto (1981). In these early works the unknown parameters in the manipulator dynamic equations, which are position-dependent quantities, are treated as constants in the stability analysis in order to show the asymptotic stability of the control scheme. Therefore the underlying assumption is that the parameter adaptation law is much faster than the manipulator dynamics.

Recent works in this field include those by Balestino et al. (1983a; 1983b), Nicosia and Tomei (1984), Craig et al. (1986), Slotine and Li (1986), Hsu et al. (1987), and Sadegh and Horowitz (1987). One trend in these recent works is to assure the stability of the overall system in spite of the nonlinear nature of the parameters in the manipulator dynamic equations. This is achieved by either of the following two methods: (1) decomposing the nonlinear parameters in the manipu-

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lator dynamic equations into the product of two quantities: one constant unknown quantity, which includes the numerical values of the masses and moments of inertia of the links and the payload and the link dimensions, the other a known nonlinear function of the manipulator structural dynamics. The nonlinear functions are then assumed to be known and calculable. The parameter adaptation law is only used to estimate the unknown constant quantities [e.g., Craig et al. (1986)]; (2) utilizing nonlinear switching parameter adaptation laws, making use of the knowledge of upper bounds of the parameters in the manipulator dynamic equations. This parameter adaptation law belongs to the class of variable structure control schemes.

In Sadegh and Horowitz (1987) it is demonstrated that by modifying the parameter adaptation law using method (1) outlined above and making the Coriolis and centripetal acceleration compensation controller a bilinear function of the joint and model reference velocities instead of a quadratic function of the joint velocities, the adaptive control scheme introduced by Horowitz and Tomizuka (1980) is globally asymptotically stable. A similar technique is presented by Slotine and Li (1986). In these control schemes only joint position and velocity feedback information is required; no joint acceleration is used, and no matrix inversion is used as in Craig et al. (1986).

The main drawback of the above-mentioned works is the computational complexity of the schemes. The schemes require on-line computations of a large amount of nonlinear functions of the joint positions and velocities. Digital implementations of such schemes may require a slow sampling rate, which in turn may deteriorate the performance of the controller.

In this paper we present a modified version of the scheme introduced in Sadegh and Horowitz (1987). The modification consists in utilizing the desired joint positions and velocities in the computation of the nonlinearity compensation controller and the parameter adaptation law instead of the actual quantities. For a given desired manipulator trajectory, the nonlinear functions of desired joint positions and velocities could be calculated and stored off-line. The modified scheme has the following advantages over the original one:

1. The amount of on-line calculations is largely reduced, and as a result, the scheme can be implemented much more efficiently.
2. Using the desired quantities in the adaptation law removes the problem of noise correlation between the estimation error and the adaptation signal (Rohrs 1982). Hence the robustness of the adaptive control law is also enhanced.
3. If the manipulator task is repetitive, the scheme can be made into a learning one. This topic will be pursued in a future article.

In this article we begin by proving additional properties of the scheme introduced by Sadegh and Horowitz (1987). These properties are (1) the global exponential stability of the overall disturbance-free system when the true parameters are used, (2) the input/output stability of (1) with respect to input disturbances, (3) the global asymptotic stability of the adaptive control scheme, and (4) sufficient conditions for the convergence of the estimated parameters to their true value for the adaptive case.

Following the above-mentioned analysis, we present a modified scheme that will be referred to as desired trajectory adaptive control. We will show that the above properties can still be preserved, provided that sufficiently large control gains are used and an explicitly defined auxiliary nonlinear feedback term is employed to compensate for the additional error introduced by the modifications.

In section 2, the manipulator's dynamic equations of motion are presented. In section 3, the manipulator control laws and their properties are presented. Section 4 contains the simulation results and discussions. Conclusions are given in section 5.

## 2. Dynamic Model of a Robotic Manipulator

In this article we consider a robotic manipulator composed of a serial open chain of rigid links connected with revolute joints. The dynamic equations of motion for the manipulator can be expressed in the following form:

$$\frac{d}{dt}\,\mathbf{x}_p(t) = \mathbf{x}_v(t) \tag{1}$$

$$\mathbf{M}(\mathbf{x}_p)\frac{d}{dt}\mathbf{x}_v(t) = \mathbf{q}(t) - \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v) - \mathbf{g}(\mathbf{x}_p) - \mathbf{c}(\mathbf{x}_p, \mathbf{x}_v)$$

where $\mathbf{x}_p$ is the $n \times 1$ vector of joint positions; $\mathbf{x}_v$ is the $n \times 1$ vector of joint velocities; $\mathbf{M}(\mathbf{x}_p)$ is the $n \times n$ symmetric and positive definite matrix (also called the generalized inertia matrix); $\mathbf{q}(t)$ is the $n \times 1$ vector of joint torques or forces supplied by the actuators; $\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v)$ is the $n \times 1$ vector resulting from Coriolis and centripetal accelerations; $\mathbf{g}(\mathbf{x}_p)$ is the $n \times 1$ vector caused by gravitational forces; and $\mathbf{c}(\mathbf{x}_p, \mathbf{x}_v)$ is the $n \times 1$ vector caused by friction forces. $\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v)$ can be expressed in the following form:

$$\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v) = \begin{bmatrix} \mathbf{x}_v{}^T\mathbf{N}^1(\mathbf{x}_p)\mathbf{x}_v \\ \mathbf{x}_v{}^T\mathbf{N}^2(\mathbf{x}_p)\mathbf{x}_v \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{x}_v{}^T\mathbf{N}^n(\mathbf{x}_p)\mathbf{x}_v \end{bmatrix}, \tag{2}$$

where the matrices $\mathbf{N}^i$'s are symmetric.

The following relation is satisfied between the matrices $\mathbf{N}^i$'s and the generalized inertia matrix $\mathbf{M}$:

$$\mathbf{N}^i(\mathbf{x}_p) = \frac{1}{2}\left[\frac{\partial \mathbf{m}_i}{\partial \mathbf{x}_p} + \left(\frac{\partial \mathbf{m}_i}{\partial \mathbf{x}_p}\right)^T - \frac{\partial \mathbf{M}}{\partial x_{pi}}\right] \tag{3}$$

where $\mathbf{m}_i$ is the $i$th column of $\mathbf{M}$, and $x_{pi}$ is the $i$th element of $\mathbf{x}_p$. The derivation of eqs. (2) and (3) is given in appendix C.

We adopt the following notations with regard to the Coriolis and centripetal acceleration vector in the remainder of this article. For $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathbf{R}^n$ let:

$$\begin{aligned} &v_i(\mathbf{x}_p, \mathbf{w}_1) = \mathbf{w}_1{}^T\,\mathbf{N}^i(\mathbf{x}_p)\,\mathbf{w}_1 \qquad \text{and} \\ &v_i(\mathbf{x}_p, \mathbf{w}_1, \mathbf{w}_2) = \mathbf{w}_1{}^T\,\mathbf{N}^i(\mathbf{x}_p)\,\mathbf{w}_2 = \mathbf{w}_2{}^T\,\mathbf{N}^i(\mathbf{x}_p)\,\mathbf{w}_1, \end{aligned} \tag{4}$$

where $\mathbf{v}_i$ is the $i$th element of $v$. The $i$th element of the friction force vector $\mathbf{c}(\mathbf{x}_p, \mathbf{x}_v)$ can be expressed as

$$c_i(x_{v_i}, q_i) = c_{ci}(x_{v_i}, q_i) + c_{li}\,x_{v_i}(t), \tag{5}$$

where $c_{ci}(x_{v_i}, q_i)$ represents the Coulomb friction component and $c_{li}\,x_{v_i}(t)$ represents the linear friction component. The $i$th components of $\mathbf{x}_v$ and $\mathbf{q}$ are denoted by $x_{v_i}$ and $q_i$, respectively.

The Coulomb friction term has a significant effect on performance of indirect robot arms. It is described by

$$c_{ci}(x_{v_i}, q_i) = \begin{cases} c_{cmi}\,\mathrm{sign}[x_{v_i}(t)] & \text{if } |x_{v_i}(t)| > 0 \\ c_{cmi}\,\mathrm{sign}[q_i(t)] & \text{if } |x_{v_i}(t)| = 0 \text{ and } |q_i(t)| > c_{cmi} \\ q_i(t) & \text{if } |x_{v_i}(t)| = 0 \text{ and } |q_i(t)| \leq c_{cmi} \end{cases} \tag{6}$$

where $c_{cmi}$ is the magnitude of the friction force.

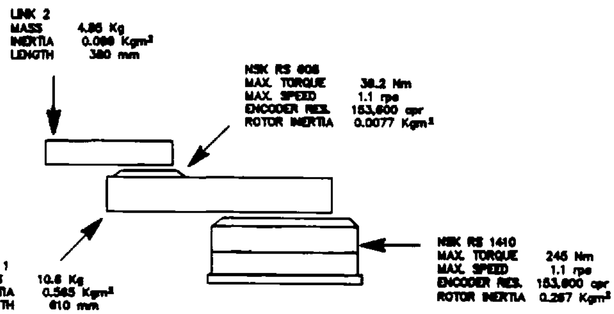
The elements of the matrices $\mathbf{M}(\mathbf{x}_p)$, $\mathbf{N}(\mathbf{x}_p)'s$ and of the vector $\mathbf{g}(\mathbf{x}_p)$ are in general nonlinear functions of the position vector $\mathbf{x}_p$. They are also a function of the link and payload masses and moments of inertia, which may not be precisely known or may vary during the manipulator task. The variation of these parameters is especially significant in the dynamic response of direct drive robotic arms.

### 2.1. Dynamic Equations Reparameterization

To dynamically control a manipulator, it is necessary to supply a torque input to the actuators that contains both a feedback and a feedforward control action. The feedback part of the control action, as will be discussed in the following section, does not require intensive on-line calculations. The feedforward part of the controller contains the reference torque input plus the nonlinearity compensation. In order to perform an exact feedforward compensation, knowledge of the elements of $\mathbf{M}(\mathbf{x}_p)$, $\mathbf{N}(\mathbf{x}_p)'s$, and $\mathbf{g}(\mathbf{x}_p)$ is required during the entire operation of the robot. These quantities, as was mentioned earlier, are nonlinear functions of joint positions and velocities. The characteristic of these nonlinear functions, however, can be determined from the kinematics of a particular robot. The coefficients of these nonlinear functions, on the other hand, are determined from the inertia characteristics of the manipulator, which may not be precisely known a priori and may be subjected to variations.

In this article we will utilize the dynamic equation reparameterization method proposed by Craig et al. (1986). In this method, eq. (1) is reparameterized into

*Fig. 1. NSK direct drive SCARA arm.*




the product of an unknown constant vector, which is a function of the unknown masses and moments of inertia of the links and the payload, and a matrix formed by known functions of $\mathbf{x}_p$, $\mathbf{x}_v$, and $\dot{\mathbf{x}}_v$, where $\dot{\mathbf{x}}_v = (d/dt)\mathbf{x}_v$ are the joint accelerations.

$$\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \dot{\mathbf{x}}_v)\boldsymbol{\Theta} = \mathbf{M}(\mathbf{x}_p)\dot{\mathbf{x}}_v + \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v) + \mathbf{g}(\mathbf{x}_p) \quad (7)$$

where $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \dot{\mathbf{x}}_v)$ is an $n \times r$ matrix and $\boldsymbol{\Theta}$ is an $r \times 1$ vector of the unknown constant parameters. Notice that $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \dot{\mathbf{x}}_v)$ is a linear function of $\dot{\mathbf{x}}_v$. We also define:

$$\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\,\boldsymbol{\Theta} = \mathbf{M}(\mathbf{x}_p)\mathbf{u} + \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v) + \mathbf{g}(\mathbf{x}_p), \quad (8)$$

where the vectors $\hat{\mathbf{x}}_v$ and $\mathbf{u}$ will be defined in section 3, and the conventions is eq. (4) are utilized in regard to the Coriolis and centripetal acceleration vector.

### 2.2. Illustrative Example

Consider a two-link SCARA manipulator moving on the horizontal plane [i.e., $\mathbf{g}(\mathbf{x}_p) = 0$] shown in Figure 1. The links are of uniform density and have lengths $l_1$ and $l_2$. The masses and moments of inertia of the links are $\mathrm{m}_1$, $\mathrm{m}_2$, $\mathbf{I}_1$, and $\mathbf{I}_2$, respectively. It can be shown that the elements of the inertia matrix and the Coriolis and centripetal acceleration vector are as follows:

$$\mathbf{M}(\mathbf{x}_p) = \begin{bmatrix} \mathbf{I}_1 + \mathbf{I}_2 + (0.25\,\mathrm{m}_1 + \mathrm{m}_2)l_1^2 + \mathrm{m}_2 l_1 l_2 \cos(x_{p_2}) & \mathbf{I}_2 + 0.25\,\mathrm{m}_2 l_2^2 + 0.5\,\mathrm{m}_2 l_1 l_2 \cos(x_{p_2}) \\ \mathbf{I}_2 + 0.25\,\mathrm{m}_2 l_2^2 + 0.5\,\mathrm{m}_2 l_1 l_2 \cos(x_{p_2}) & \mathbf{I}_2 + 0.25\,\mathrm{m}_2 l_2^2 \end{bmatrix} \quad (9)$$

$$\mathbf{v}^T(\mathbf{x}_p, \mathbf{x}_v) = \begin{bmatrix} (-\mathrm{m}_2 l_1 l_2 x_{v_1} x_{v_2} + 0.5\,\mathrm{m}_2 l_1 l_2 x_{v_2} x_{v_2}) \sin(x_{p_2}) & 0.5\,\mathrm{m}_2 l_1 l_2 x_{v_1} x_{v_1} \sin(x_{p_2}). \end{bmatrix} \quad (10)$$

Defining

$$\boldsymbol{\Theta}_1 = \mathbf{I}_1 + \mathbf{I}_2 + (0.25\,\mathrm{m}_1 + \mathrm{m}_2)l_1^2 \quad (11)$$

$$\boldsymbol{\Theta}_2 = \mathbf{I}_2 + 0.25\,\mathrm{m}_2 l_2^2 \quad \text{and} \quad (12)$$

$$\boldsymbol{\Theta}_3 = 0.5\,\mathrm{m}_2 l_1 l_2, \quad (13)$$

and utilizing eq. (8), the following expression for the $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})$ matrix is obtained

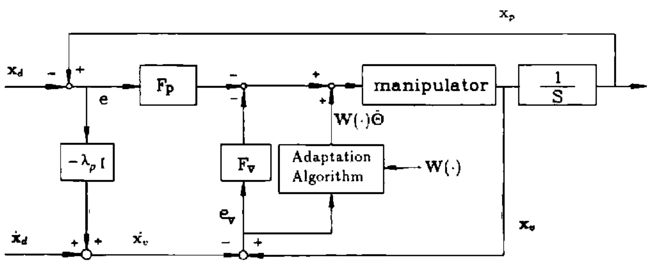
$$\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u}) = \begin{bmatrix} u_1 & u_2 & 2u_1 \cos(x_{p_2}) + u_2 \cos(x_{p_2}) - (x_{v_2}\hat{x}_{v_2} + x_{v_1}\hat{x}_{v_2} + \hat{x}_{v_1} x_{v_2}) \sin(x_{p_2}) \\ 0 & u_1 + u_2 & u_1 \cos(x_{p_2}) + x_{v_1}\hat{x}_{v_1} \sin(x_{p_2}) \end{bmatrix} \quad (14)$$

## 3. Manipulator Control Law and Its Properties

The control objective is to force the manipulator to track a set of given joint positions and velocities with desirable dynamics. Let us denote the desired quantities by $\mathbf{x}_d(t)$ and $\dot{\mathbf{x}}_d(t)$, respectively. Assume $\dot{\mathbf{x}}_d(t)$ is differentiable, and denote its derivative by $\ddot{\mathbf{x}}_d(t)$. $\ddot{\mathbf{x}}_d(t)$ is referred to as the desired acceleration.

We first review the control scheme in Slotine and Li (1986) and Sadegh and Horowitz (1987), and introduce some additional properties of the control scheme. This scheme employs an exact compensation for all the nonlinearities in the manipulator dynamics.

*Fig. 2. Overall control system block diagram.*




### 3.1. Exact Compensation Control Law (ECCL)

*Fixed Controller*

The control algorithm is a fixed proportional plus derivative (PD) controller plus a feedforward inertia, Coriolis and centripital acceleration and gravity compensator. The control task is performed by employing an inner velocity and an outer position feedback loop. The inner velocity feedback loop contains the adaptation law, if desired. The outer feedback loop is a fixed proportional position feedback.

Let $\mathbf{e}(t)$ denote the tracking error:

$$\mathbf{e}(t) = \mathbf{x}_p(t) - \mathbf{x}_d(t) \tag{15}$$

The model reference velocity for the inner loop is obtained from the following linear dynamics:

$$\frac{d}{dt}\hat{\mathbf{x}}_v(t) = \mathbf{u}(t), \tag{16}$$

$$\mathbf{u}(t) = \ddot{\mathbf{x}}_d(t) - \lambda_p \dot{\mathbf{e}}(t), \qquad \lambda_p > 0. \tag{17}$$

Defining the error corresponding to the reference and actual velocity to be:

$$\mathbf{e}_v(t) = \mathbf{x}_v(t) - \hat{\mathbf{x}}_v(t), \tag{18}$$

the control law is given by:

$$\mathbf{q}(t) = \mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\hat{\mathbf{\Theta}}(t) - \mathbf{F}_v\mathbf{e}_v(t) - \mathbf{F}_p\mathbf{e}(t) + \mathbf{c}(\mathbf{x}_p, \mathbf{x}_v), \tag{19}$$

where $\mathbf{F}_p = \sigma_p\mathbf{I}$ and $\mathbf{F}_v = \sigma_v\mathbf{I}$ are constant diagonal positive definite gain matrices ($\sigma_p, \sigma_v > 0$) and $\hat{\mathbf{\Theta}}(t)$ is an estimate of $\mathbf{\Theta}$. (The diagonal structure of the gain matrices is chosen for simplicity and poses no loss of generality in the subsequent stability analysis.)

Employing the control law given by (19) in eq. (1) we obtain the following error dynamics equation:

$$\begin{aligned} \mathbf{M}(\mathbf{x}_p)\dot{\mathbf{e}}_v + \mathbf{v}(\mathbf{x}_p\, \mathbf{x}_v, \mathbf{e}_v) + \mathbf{F}_v\mathbf{e}_v + \mathbf{F}_p\mathbf{e} = \mathbf{w}, \\ \dot{\mathbf{e}} = -\lambda_p\mathbf{e} + \mathbf{e}_v, \end{aligned} \tag{20}$$

where

$$\mathbf{w} = \mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\tilde{\mathbf{\Theta}} + \mathbf{w}_d, \qquad \tilde{\mathbf{\Theta}} = \hat{\mathbf{\Theta}} - \mathbf{\Theta}. \tag{21}$$

$\mathbf{w}_d$ is the sum of other possible inputs and/or disturbances to the manipulator.

*Remark (i)* In most of the early works on manipulator adaptive control, the error dynamics is forced to behave as a stable, linear, time-invariant system. This is achieved by cancelling the Coriolis and centripetal acceleration term, $\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v)$, without properly accounting for the fact that this term and the time derivative of the manipulator kinetic energy are related as follows [e.g., see Koditschek (1984) and Arimoto and Miyazaki (1984)]:

$$\frac{1}{2}\frac{d}{dt}[\mathbf{x}_v{}^T\mathbf{M}(\mathbf{x}_p)\mathbf{x}_v] = \frac{d}{dt}\mathbf{x}_v{}^T\mathbf{M}(\mathbf{x}_p)\mathbf{x}_v + \frac{1}{2}\mathbf{x}_v{}^T\frac{d}{dt}\mathbf{M}(\mathbf{x}_p)\,\mathbf{x}_v$$

and

$$\frac{1}{2}\mathbf{x}_v{}^T\frac{d}{dt}\mathbf{M}(\mathbf{x}_p)\,\mathbf{x}_v - \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v) = 0$$

As a result, one is forced to assume in the stability analysis that the rate of change of the inertia matrix is negligible as compared to the parameter adaptation law response [i.e., $(d/dt)\mathbf{M}(\mathbf{x}_p) \approx 0$].

*Remark (ii)* In the approach pursued here, the error dynamics given by eq. (20) is nonlinear. As will be seen in the following theorem, the term $\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{e}_v)$, whose structure is similar to the Coriolis and centripetal acceleration term, cancels the effect of the inertia matrix time derivative, $(d/dt)\,\mathbf{M}(\mathbf{x}_p)$, in the Lyapunov analysis. See also Slotine and Li (1986).

To summarize the properties of the error eq. (20), we present the following theorem.

Theorem 1

Consider the error system given by eq. (20); then:

The undisturbed (i.e., $\mathbf{w} = 0$) closed-loop error system is globally exponentially stable [i.e., both $\mathbf{e}_v(t)$ and $\mathbf{e}(t)$ converge to zero exponentially from a given initial condition]. For definitions see Appendix (1).

PROOF

The proof of this theorem is based on the Lyapunov approach. Let us first define a generalized error state vector to be

$$\bar{\mathbf{e}} := [\mathbf{e}_v\ \mathbf{e}]^T, \tag{22}$$

and the maximum and minimum eigenvalues of $\mathbf{M}$ to be

$$\lambda_m := \inf_{\mathbf{x}_p} \inf_{|\mathbf{w}|=1} |\mathbf{M}(\mathbf{x}_p)\mathbf{w}| \tag{23}$$

$$\lambda_M := \sup_{\mathbf{x}_p} \sup_{|\mathbf{w}|=1} |\mathbf{M}(\mathbf{x}_p)\mathbf{w}|.$$

Define the Lyapunov function candidate as

$$V(t, \bar{\mathbf{e}}) = \frac{1}{2}\, \mathbf{e}_v^T\, \mathbf{M}(\mathbf{e} + \mathbf{x}_d(t))\, \mathbf{e}_v + \frac{1}{2}\, \mathbf{e}^T\, \mathbf{F}_p\, \mathbf{e} \tag{24}$$

$V(t, \bar{\mathbf{e}})$ is a legitimate Lyapunov function candidate since

$$\frac{1}{2} \min(\lambda_m, \sigma_p)\, |\bar{\mathbf{e}}|^2 \leq V(t, \bar{\mathbf{e}}) \leq \frac{1}{2} \max(\lambda_M, \sigma_p)\, |\bar{\mathbf{e}}|^2.$$

From (20) and (24) we obtain

$$\begin{aligned} \frac{d}{dt} V(t, \bar{\mathbf{e}}) &= \mathbf{e}_v^T(-\mathbf{F}_v\mathbf{e}_v - \mathbf{F}_p\mathbf{e}) - \mathbf{e}_v^T\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{e}_v) \\ &\quad + \frac{1}{2}\, \mathbf{e}_v^T\dot{\mathbf{M}}(\mathbf{x}_p)\mathbf{e}_v \\ &\quad + \mathbf{e}^T\mathbf{F}_p(-\lambda_p\mathbf{e} + \mathbf{e}_v) \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{d}{dt} V(t, \bar{\mathbf{e}}) &= -\mathbf{e}_v^T\, \mathbf{F}_v\, \mathbf{e}_v - \lambda_p \mathbf{e}^T\, \mathbf{F}_p\, \mathbf{e} \\ &\quad - \mathbf{e}_v^T\left[\frac{1}{2}\, \dot{\mathbf{M}}(\mathbf{x}_p)\mathbf{e}_v - \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{e}_v)\right] \end{aligned} \tag{26}$$

Rewriting the matrix $\mathbf{M}$ as

$$\mathbf{M} = [\mathbf{m}_1\ \mathbf{m}_2\ \cdots\ \mathbf{m}_n],$$

then

$$\begin{aligned} \dot{\mathbf{M}}(\mathbf{x}_p)\mathbf{e}_v &= \left[\frac{\partial \mathbf{M}(\mathbf{x}_p)}{\partial x_{p_1}}\, \mathbf{e}_v\ \cdots\ \frac{\partial \mathbf{M}(\mathbf{x}_p)}{\partial x_{p_n}}\, \mathbf{e}_v\right] \mathbf{x}_v \\ &= \left[\mathbf{x}_v^T\left[\frac{\partial \mathbf{m}_1(\mathbf{x}_p)}{\partial \mathbf{x}_p}\right]^T \mathbf{e}_v\ \cdots\ \mathbf{x}_v^T\left[\frac{\partial \mathbf{m}_n(\mathbf{x}_p)}{\partial \mathbf{x}_p}\right]^T \mathbf{e}_v\right]^T. \end{aligned} \tag{27}$$

Notice that from eq. (27) and eq. (3), the third term in eq. (26),

$$\mathbf{e}_v^T\left[\frac{1}{2}\, \dot{\mathbf{M}}(\mathbf{x}_p)\, \mathbf{e}_v - \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{e}_v)\right] = -\mathbf{e}_v^T\mathbf{R}(\mathbf{x}_p, \mathbf{x}_v)\mathbf{e}_v = 0,$$

since the matrix

$$\mathbf{R}(\mathbf{x}_p, \mathbf{x}_v) = \frac{1}{2}\begin{bmatrix} \mathbf{x}_v^T\left(\dfrac{\partial \mathbf{m}_1}{\partial \mathbf{x}_p} - \dfrac{\partial \mathbf{M}}{\partial x_{p_1}}\right) \\ \mathbf{x}_v^T\left(\dfrac{\partial \mathbf{m}_2}{\partial \mathbf{x}_p} - \dfrac{\partial \mathbf{M}}{\partial x_{p_2}}\right) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{x}_v^T\left(\dfrac{\partial \mathbf{m}_n}{\partial \mathbf{x}_p} - \dfrac{\partial \mathbf{M}}{\partial x_{p_n}}\right) \end{bmatrix}$$

is skew symmetric.

$$\begin{aligned} \frac{d}{dt} V(t, \bar{\mathbf{e}}) &= -\mathbf{e}_v^T\, \mathbf{F}_v\, \mathbf{e}_v - \lambda_p\, \mathbf{e}^T\, \mathbf{F}_p\, \mathbf{e} \\ &= -\sigma_v|\mathbf{e}_v|^2 - \lambda_p\, \sigma_p\, |\mathbf{e}|^2, \end{aligned} \tag{28}$$

$$\frac{d}{dt} V(t, \bar{\mathbf{e}}) \leq -\gamma V(t, \bar{\mathbf{e}}) \leq 0, \tag{29}$$

where

$$\gamma = \min(\sigma_v/\lambda_M, \lambda_p) \tag{30}$$

Multiplying both sides of (29) by $e^{\gamma t}$ and integrating, we obtain

$$V(t, \bar{\mathbf{e}}) \leqslant e^{-\gamma t} V[0, \bar{\mathbf{e}}(0)] \tag{31}$$

Thus

$$|\mathbf{e}_v(t)| \leqslant \frac{1}{\sqrt{\lambda_m}} e^{-(\gamma/2)t} V^{1/2}[0, \bar{\mathbf{e}}(0)] \tag{32}$$

$$|\mathbf{e}(t)| \leqslant \frac{1}{\sqrt{\sigma_p}} e^{-(\gamma/2)t} V^{1/2}[0, \bar{\mathbf{e}}(0)]. \tag{33}$$

Thus both $\mathbf{e}_v(t)$ and $\mathbf{e}(t)$ converge to zero exponentially. □

Corollary 1

The perturbed system (i.e., $\mathbf{w}_d \neq 0$) is $\mathbf{L}_p$ input/output stable with respect to the pairs $(\mathbf{w}_d, \mathbf{e}_v)$ and $(\mathbf{w}_d, \mathbf{e})$ for all $p \in [1, \infty]$, i.e., there exist positive constants $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ such that:

$$\|\mathbf{e}_v(.)\|_p \leqslant \alpha_1 \|\mathbf{w}_d(.)\|_p + \alpha_2, \tag{34}$$

$$\|\mathbf{e}(.)\|_p \leqslant \beta_1 \|\mathbf{w}_d(.)\|_p + \beta_2, \tag{35}$$

For definitions see Appendix A.

PROOF

In theorem (1) we proved that the undisturbed system is globally exponentially stable. Following the lines of Vidyasagar and Vannelli (1982), we can show that the disturbance driven system is input/output stable.

If $\mathbf{w}_d \neq 0$, eq. (28) becomes

$$\begin{aligned} \frac{d}{dt} V &= -\mathbf{e}_v{}^T \mathbf{F}_v \mathbf{e}_v - \lambda_p \mathbf{e}^T \mathbf{F}_p \mathbf{e} + \mathbf{e}_v{}^T \mathbf{w}_d \\ &\leqslant -\gamma V + \frac{1}{\sqrt{\lambda_m}} V^{1/2} |\mathbf{w}_d|, \end{aligned} \tag{36}$$

where arguments of $V$ have been dropped for economy.

$V^{1/2}$ is differentiable except at $(\mathbf{e}_v, \mathbf{e}) = 0$. For time intervals in which $(\bar{\mathbf{e}}) = 0$, the Lyapunov function is identically equal to zero. Therefore we consider the time intervals that $(\bar{\mathbf{e}}) \neq 0$. Dividing eq. (36) by $V^{1/2} \neq 0$, we obtain

$$\frac{d}{dt} V^{1/2} + \frac{\gamma}{2} V^{1/2} \leqslant \frac{1}{2\sqrt{\lambda_m}} |\mathbf{w}_d|. \tag{37}$$

Multiplying both sides of inequality (37) by $e^{(\gamma/2)t}$ and integrating, we obtain

$$V^{1/2}(t, \bar{\mathbf{e}}) \leqslant e^{-(\gamma/2)t} V^{1/2}(0, \bar{\mathbf{e}}(0)) + V_d, \tag{38}$$

where

$$V_d = \frac{1}{2\sqrt{\lambda_m}} \int_0^t e^{-(\gamma/2)(t-\tau)} |\mathbf{w}_d(\tau)| d\tau. \tag{39}$$

The term $V_d$ in eq. (38) is a convolution operator on $|\mathbf{w}_d|$, and from well-known results in linear system theory [see Desoer and Vidyasagar (1975)], we obtain

$$|V_d(.)|_p \leqslant \frac{1}{\gamma\sqrt{\lambda_m}} |\mathbf{w}_d(.)|_p, \tag{40}$$

for all $p \in [1, \infty]$.

Thus,

$$\|\mathbf{e}_v(.)\|_p \leqslant \alpha_2 + \alpha_1 \|\mathbf{w}_d(.)\|_p \tag{41}$$

$$\|\mathbf{e}(.)\|_p \leqslant \beta_2 + \beta_1 \|\mathbf{w}_d(.)\|_p, \tag{42}$$

where

$$\alpha_1 = \frac{1}{\gamma \lambda_m} \qquad \alpha_2 = \frac{2}{\gamma\sqrt{\lambda_m}} V^{1/2}(0, \bar{\mathbf{e}}(0)) \tag{43}$$

$$\beta_1 = \frac{1}{\gamma\sqrt{\sigma_p \lambda_m}} \qquad \beta_2 = \frac{2}{\gamma\sqrt{\sigma_p}} V^{1/2}(0, \bar{\mathbf{e}}(0)) \tag{44}$$

□

*Exact Compensation Adaptation Law (ECAL)*

To estimate the parameter vector $\Theta$, the following adaptation law is utilized.

$$\frac{d}{dt} \hat{\mathbf{\Theta}}(t) = -\mathbf{K}\, \mathbf{W}^T(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\, \mathbf{e}_v(t), \qquad \mathbf{K} > 0 \tag{45}$$

Notice that in the parameter adaptation algorithm (PAA), only the velocity error signal $\mathbf{e}_v(t)$ is used. Also, comparing with the method presented by Craig et al. (1986), the acceleration input $\mathbf{u}(t)$ is used in the PAAs instead of the joint accelerations $\dot{\mathbf{x}}_v$, which are not measurable in most realistic applications, and no matrix inversion is required in the control algorithm.

Theorem 2

For an error dynamics governed by eq. (22) with $\mathbf{w}_d = 0$, if the adaptive control law given (45) is used, then the following properties hold:

(*i*). The error between the desired and actual trajectory converges asymptotically to zero, i.e.,

$$\lim_{t\to\infty} [\mathbf{x}_d(t) - \mathbf{x}_p(t)] = 0 \quad \text{and}$$

$$\lim_{t\to\infty} [\dot{\mathbf{x}}_d(t) - \mathbf{x}_v(t)] = 0$$

(*ii*). If $\ddot{\mathbf{x}}_d(t)$ is uniformly continuous almost everywhere, (for definition see Appendix A), then:
  (a). $\lim_{t\to\infty} \mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\dot{\mathbf{x}}}_v, \mathbf{u})\,\tilde{\boldsymbol{\Theta}} = 0$.
  (b). If, in addition, $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ is persistently exciting, then $\lim_{t\to\infty} \tilde{\boldsymbol{\Theta}}(t) = 0$

$\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ is defined by eq. (7), except that $\mathbf{x}_p$, $\mathbf{x}_v$, and $\dot{\mathbf{x}}_v$ are replaced by their desired counterparts.

The definition of persistent excitation is given in Appendix A.

PROOF

(*i*)

Consider the system described by the error equation eq. (20) and the PAA eq. (45). Define the Lyapunov function candidate

$$V(t, \bar{\mathbf{e}}) = \frac{1}{2}\mathbf{e}_v^T M(\mathbf{e} + \mathbf{x}_d(t))\,\mathbf{e}_v + \frac{1}{2}\mathbf{e}^T \mathbf{F}_p\,\mathbf{e} + \frac{1}{2}\tilde{\boldsymbol{\Theta}}^T \mathbf{K}^{-1}\tilde{\boldsymbol{\Theta}}. \quad (46)$$

In a similar fashion as in the proof of theorem (1) we obtain:

$$\frac{d}{dt}V(t, \bar{\mathbf{e}}) = -\mathbf{e}_v^T \mathbf{F}_v\,\mathbf{e}_v - \lambda_p\,\mathbf{e}^T \mathbf{F}_p\,\mathbf{e} + \mathbf{e}_v^T \mathbf{w}_d + \mathbf{e}_v^T \mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\dot{\mathbf{x}}}_v, \mathbf{u})\,\tilde{\boldsymbol{\Theta}}(t) + \tilde{\boldsymbol{\Theta}}^T(t)\,\mathbf{K}^{-1}\frac{d}{dt}\tilde{\boldsymbol{\Theta}}(t) \quad (47)$$

From the parameter adaptation law, eq. (45), we obtain:

$$\frac{d}{dt}V(t, \bar{\mathbf{e}}) = -\mathbf{e}_v^T \mathbf{F}_v\,\mathbf{e}_v - \lambda_p\,\mathbf{e}^T \mathbf{F}_p\,\mathbf{e} \quad (48)$$

Therefore $|\mathbf{e}_v|$, $|\mathbf{e}|$ and $|\tilde{\boldsymbol{\Theta}}|$ are all bounded. Consequently $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\dot{\mathbf{x}}}_v, \mathbf{u})\,\tilde{\boldsymbol{\Theta}}(t)$ is bounded and, from standard adaptive control arguments [Narendra and Valvani (1980)], the asymptotic convergence of $|\mathbf{e}_v(t)|$, $|\mathbf{e}(t)|$ and $|\dot{\mathbf{e}}(t)|$ follows immediately. □

The proof of (*ii*) is in Appendix B.

### 3.2. Desired Compensation Control Law (DCCL)

In order to implement the adaptive controller described above, one needs to calculate the elements of $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\dot{\mathbf{x}}}_v, \mathbf{u})$ in real time. As can be seen in the example above, this procedure may be excessively time consuming since it involves computations of highly nonlinear functions of joint position and velocities. Consequently the real-time implementation of such a scheme is rather difficult. To overcome this difficulty, we suggest replacing $\mathbf{x}_p$ and $\mathbf{x}_d$ with their desired counterparts, namely $\mathbf{x}_d$ and $\dot{\mathbf{x}}_d$, in the $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \dot{\mathbf{x}}_v)$ matrix defined by eq. (7).

The desired quantities are known in advance, and therefore all their corresponding calculations can be performed off-line. Thus the real-time implementation of the scheme becomes more feasible.

Our task is now to show that the above stability properties can still be preserved.

The modified control law is given by

$$\mathbf{q}(t) = \mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\hat{\boldsymbol{\Theta}} - \mathbf{F}_v\mathbf{e}_v - \mathbf{F}_p\mathbf{e} + \mathbf{c}(\mathbf{x}_p, \mathbf{x}_v) - \mathbf{f}(\mathbf{e}_v, \mathbf{e}), \quad (49)$$

where

$$\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\boldsymbol{\Theta} = \mathbf{M}(\mathbf{x}_d)\ddot{\mathbf{x}}_d + \mathbf{v}(\mathbf{x}_d, \dot{\mathbf{x}}_d) + \mathbf{g}(\mathbf{x}_d), \quad (50)$$

and $\hat{\boldsymbol{\Theta}}$ is an estimate of $\boldsymbol{\Theta}$. Also, $\mathbf{f}(\mathbf{e}_v, \mathbf{e})$ is an auxiliary nonlinear feedback term with a constant gain to be defined later. The role of this additional term is to compensate for the additional error introduced by the modification of the original adaptive controller.

Applying the control law given by eq. (49) to the manipulator described by eq. (1), we obtain the following error dynamic equation:

$$
\begin{gathered}
\mathbf{M}(\mathbf{x}_p)\dot{\mathbf{e}}_v + \mathbf{v}(\mathbf{x}_v, \mathbf{x}_p, \mathbf{e}_v) + \mathbf{F}_v\mathbf{e}_v + \mathbf{F}_p\mathbf{e} + \Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) + \mathbf{f} = \mathbf{w} \\
\dot{\mathbf{e}} = -\lambda_p\mathbf{e} + \mathbf{e}_v
\end{gathered}
\tag{51}
$$

where

$$
\mathbf{w} = \mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\,\tilde{\mathbf{\Theta}} + \mathbf{w}_d, \tag{52}
$$

$$
\begin{aligned}
\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) = (\,&\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u}) \\
&- \mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d))\mathbf{\Theta},
\end{aligned}
\tag{53}
$$

and $\mathbf{w}_d$ is as defined previously.

As can be seen in equation (51), an additional error $\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e})$ is introduced. The following lemma provides explicit bounds on $\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e})$.

### Lemma 1

For the error system governed by eq. (51) the following inequality holds:

$$
\begin{aligned}
-\mathbf{e}_v^T\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) \leqslant\ &\mathbf{e}_v^T(\lambda_p\mathbf{M}(\mathbf{x}_p) + \mathrm{b}_1\mathbf{I})\mathbf{e}_v \\
&+ \mathbf{e}_v^T(-\lambda_p^2\mathbf{M}(\mathbf{x}_p) + \mathrm{b}_2\mathbf{I})\mathbf{e} \\
&+ \mathrm{b}_3(|\mathbf{e}_v|^2|\mathbf{e}| + \lambda_p|\mathbf{e}_v|\,|\mathbf{e}|^2),
\end{aligned}
\tag{54}
$$

where $\mathrm{b}_1$, $\mathrm{b}_2$, and $\mathrm{b}_3$ are all positive valued functions that are bounded by the norm of their arguments. Explicit relations are found in Appendix 1.

### Proof

See Appendix 1.

We now present the equivalent of theorem (1) for the controller with desired compensation.

### Theorem 3

For the error system governed by eq. (51), the results of theorem (1) hold, provided:

i. $\mathbf{f}(\mathbf{e}_v, \mathbf{e})$ is a nonlinear feedback given by

$$
\mathbf{f}(\mathbf{e}_v, \mathbf{e}) = \sigma_n|\mathbf{e}|^2\,\mathbf{e}_v, \quad \sigma_n > 0 \quad \text{and} \tag{55}
$$

ii. $\sigma_v$, $\sigma_p$, and $\sigma_n$ are chosen sufficiently large.

### Proof

The proof is very similar to the proof of the theorem 1, except that in this case we choose a slightly different Lyapunov function.

Choosing the Lyapunov function candidate

$$
\begin{aligned}
V(t, \bar{\mathbf{e}}) = &\frac{1}{2}\,\mathbf{e}_v^T\,\mathbf{M}(\mathbf{e} + \mathbf{x}_d(t))\,\mathbf{e}_v \\
&+ \frac{1}{2}\,\mathbf{e}^T(\mathbf{F}_p + \lambda_p^2\bar{\lambda}\mathbf{I})\,\mathbf{e}
\end{aligned}
\tag{56}
$$

where

$$
\bar{\lambda} = \frac{1}{2}(\lambda_m + \lambda_M), \tag{57}
$$

then similarly to the proof of the theorem (1), we have:

$$
\begin{aligned}
\frac{d}{dt}V(t, \bar{\mathbf{e}}) = &-\sigma_v|\mathbf{e}_v|^2 - \lambda_p(\sigma_p + \lambda_p^2\bar{\lambda})|\mathbf{e}|^2 \\
&+ \lambda_p^2\bar{\lambda}\,\mathbf{e}_v^T\mathbf{e} - \mathbf{e}_v^T\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) - \mathbf{e}_{vT}\mathbf{f}
\end{aligned}
\tag{58}
$$

By using the expression for $-\mathbf{e}_v^T\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e})$ from lemma 1 and the definition of $\mathbf{f}$ in (55), we obtain

$$
\begin{aligned}
\frac{d}{dt}V(t, \bar{\mathbf{e}}) \leqslant &-\left[\sigma_v - \lambda_p\lambda_M - \frac{\mathrm{b}_3 + 4\mathrm{b}_1}{4}\right]|\mathbf{e}_v|^2 \\
&- \lambda_p\left[\sigma_p + \lambda_p^2\bar{\lambda} - \frac{\mathrm{b}_3}{4}\right]|\mathbf{e}|^2 \\
&- \lambda_p^2\mathbf{e}_v^T[\mathbf{M} - \bar{\lambda}\mathbf{I}]\mathbf{e} + \mathrm{b}_2|\mathbf{e}_v|\,|\mathbf{e}| - \mathrm{b}_3|\mathbf{e}_v|^2\left[\frac{1}{2} - |\mathbf{e}|\right]^2 \\
&- \lambda_p\mathrm{b}_3|\mathbf{e}|^2\left[\frac{1}{2} - |\mathbf{e}_v|\right]^2 - [\sigma_n - (1 + \lambda_p)\mathrm{b}_3][|\mathbf{e}_v|\,|\mathbf{e}|]^2.
\end{aligned}
\tag{59}
$$

Notice that $|\mathbf{M} - \bar{\lambda}\mathbf{I}| \leqslant [(\lambda_M - \lambda_m)/2]$, and by choosing $\sigma_n \geqslant (1 + \lambda_p)\mathrm{b}_3$, we obtain

$$
\frac{d}{dt}V(t, \bar{\mathbf{e}}) \leqslant [|\mathbf{e}_v|\,|\mathbf{e}|]\,\mathbf{Q}\begin{bmatrix}|\mathbf{e}_v| \\ |\mathbf{e}|\end{bmatrix}, \tag{60}
$$

where

$$
\mathbf{Q} = -\begin{bmatrix}
\sigma_v - \lambda_p^2\lambda_M - \dfrac{\mathrm{b}_3 + 4\mathrm{b}_1}{4} & \dfrac{\mathrm{b}_2}{2} + \dfrac{\lambda_p^2(\lambda_M - \lambda_m)}{4} \\
\dfrac{\mathrm{b}_2}{2} + \dfrac{\lambda_p^2(\lambda_M - \lambda_m)}{4} & \lambda_p\left[\sigma_p + \lambda_p^2\bar{\lambda} - \dfrac{\mathrm{b}_3}{4}\right]
\end{bmatrix}. \tag{61}
$$

Fig. 3. *Position response of axis 2; the ECAL and DCAL (case i). --- desired trajectory; — actual trajectory.*

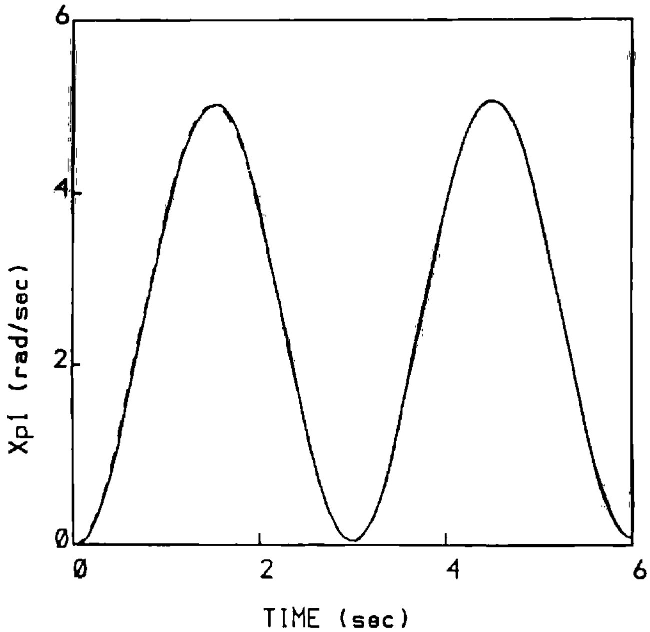
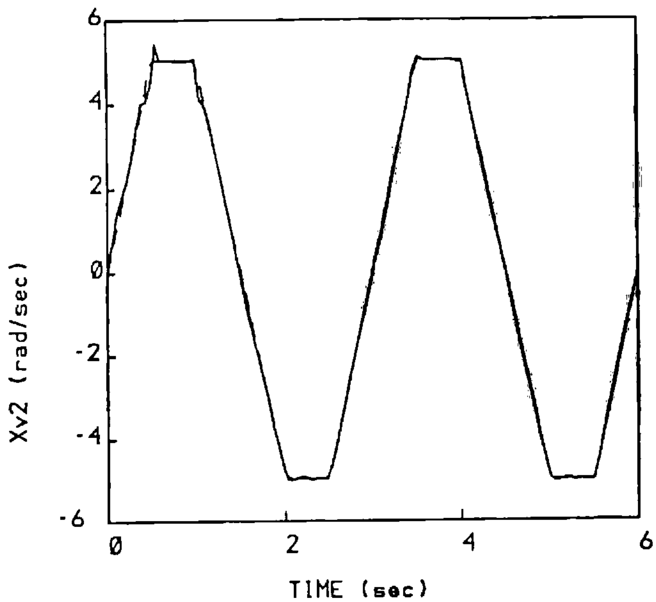



Fig. 4. *Velocity response of axis 2; the ECAL and the DCAL (case i). --- desired trajectory; — actual trajectory.*




Let us decompose $\sigma_v$ and $\sigma_p$ into:

$$\sigma_p = \bar{\sigma}_p + \tilde{\sigma}_p \quad \text{and} \quad \sigma_v = \bar{\sigma}_v + \tilde{\sigma}_v. \tag{62}$$

Consequently, $\mathbf{Q}$ can be decomposed into

$$\mathbf{Q} = -\begin{bmatrix} \bar{\sigma}_v & 0 \\ 0 & \bar{\sigma}_p \end{bmatrix} \tag{63}$$

where

$$\tilde{\mathbf{Q}} = -\begin{bmatrix} \tilde{\sigma}_v - \lambda_p{}^2\lambda_M - \frac{b_3 + 4b_1}{4} & \frac{b_2}{2} + \frac{\lambda_p{}^2(\lambda_M - \lambda_m)}{4} \\ \frac{b_2}{2} + \frac{\lambda_p{}^2(\lambda_M - \lambda_m)}{4} & \lambda_p\left[\tilde{\sigma}_p + \lambda_p{}^2\bar{\lambda} - \frac{b_3}{4}\right] \end{bmatrix}. \tag{64}$$

It is always possible to choose $\tilde{\sigma}_v$, $\tilde{\sigma}_p$, and $\lambda_p$ such that $\tilde{\mathbf{Q}} \geqslant 0$. Thus,

$$\frac{d}{dt} V(t, \bar{\mathbf{e}}) \leqslant -\bar{\sigma}_v|\mathbf{e}_v|^2 - \lambda_p\,\bar{\sigma}_p|\mathbf{e}|^2. \tag{65}$$

From this point on the proof of the theorem becomes identical to the proof of theorem 1. To obtain the explicit quantities obtained in theorem 1, simply replace $\sigma_v$ and $\sigma_p$ by $\bar{\sigma}_v$ and $\bar{\sigma}_p$. □

Corollary 2

The perturbed system (i.e., $\mathbf{w} \neq 0$) is $\mathbf{L}_p$ input/output stable with respect to the pairs $(\mathbf{w}, \mathbf{e}_v)$ and $(\mathbf{w}, \mathbf{e})$ for all $p \in [1, \infty]$.

*Remark:* Corollary 2 states that if our estimation of the manipulator dynamics is not exact, the norm of the resulting error will be proportional to the parameter error norm and norm of other possible disturbances. Therefore it is an improvement over corollary 1, which allows only for bounded disturbances.

Proof

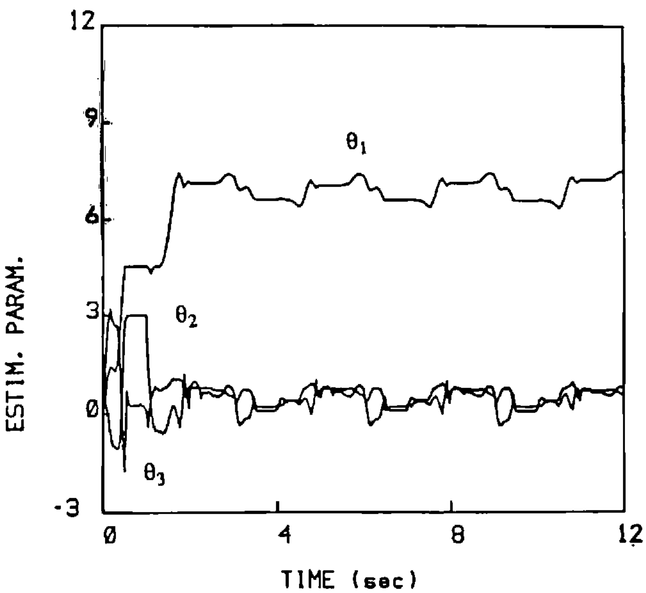
Replace eq. (28) in theorem 1 by eq. (65). The remainder of the proof is identical to the proof of corollary 1. □

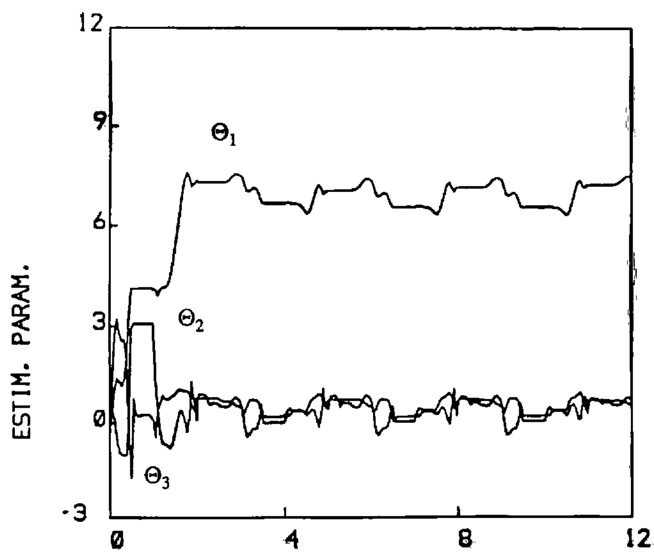
*Desired Compensation Adaptation Law (DCAL)*

The parameter update law is similar to the ECAL except for the replacement of $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})$ by

*Fig. 5. Parameter estimates for the ECAL (case* i*).*

*Fig. 6. Parameter estimates for the DCAL (case* i*).*




$\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$. Thus,

$$\frac{d}{dt}\hat{\boldsymbol{\theta}}(t) = -\mathbf{K}\,\mathbf{W}^T(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\,\mathbf{e}_v(t), \quad \mathbf{K} > 0 \tag{66}$$

Notice that the parameter adaptation law only contains the desired trajectory quantities in its adaptation signal. Therefore the adaptation signal $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ may be calculated and stored off-line. Additionally, the adaptation law in eq. (66) is more robust to stochastic disturbances than that of eq. (45). This is a result of the fact that the matrix $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ is not contaminated with noise, hence avoiding noise correlation between the error signal and the adaptation signal.

Theorem 4

For an error dynamics governed by eq. (51) with $\mathbf{w}_d = 0$, if the conditions of theorem 3 are satisfied and the adaptive control law given by eq. (66) is used, then the results of theorem 2 hold.

PROOF

Define the Lyapunov function candidate

$$V(t, \bar{\mathbf{e}}) = \frac{1}{2}\,[\mathbf{e}_v^T\,\mathbf{M}(\mathbf{e} + \mathbf{x}_d(t))\,\mathbf{e}_v + \mathbf{e}^T\,(F_p + \lambda_p^2\bar{\lambda}\mathbf{I})\,\mathbf{e} + \tilde{\boldsymbol{\theta}}^T\,\mathbf{K}^{-1}\,\tilde{\boldsymbol{\theta}}]. \tag{67}$$

Similarly to the proofs of theorems 2 and 3, we obtain:

$$\frac{d}{dt}V(t, \bar{\mathbf{e}}) \leq -\bar{\sigma}_v|\mathbf{e}_v|^2 - \lambda_p\,\bar{\sigma}_p|\mathbf{e}|^2 + \mathbf{e}_v^T\,\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T\,\mathbf{K}^{-1}\,\frac{d}{dt}\tilde{\boldsymbol{\theta}} \tag{68}$$

From the parameter adaptation law, eq. (66), we obtain:

$$\frac{d}{dt}V(t, \bar{\mathbf{e}}) \leq -\bar{\sigma}_v|\mathbf{e}_v|^2 - \lambda_p\,\bar{\sigma}_p|\mathbf{e}|^2. \tag{69}$$

The rest of the proof is identical to the proof of theorem 3. □

## 4. Simulation

Simulation studies were conducted for a two-degree-of-freedom scara robot arm. The parameters of the robot model used in the simulations are the ones of the NSK robot at the University of California-Berkeley Department of Mechanical Engineering Robotics Laboratory (Fig. 1). Refer to section 2 for the manipulator's dynamic equation and its reparameterization.

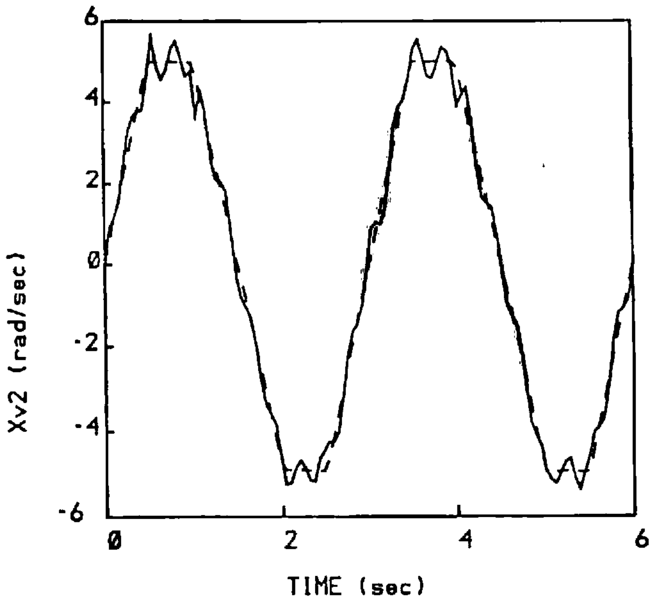



*Fig. 7. Velocity response of axis 2 for the ECAL (case* ii). *---desired trajectory; —actual trajectory.*

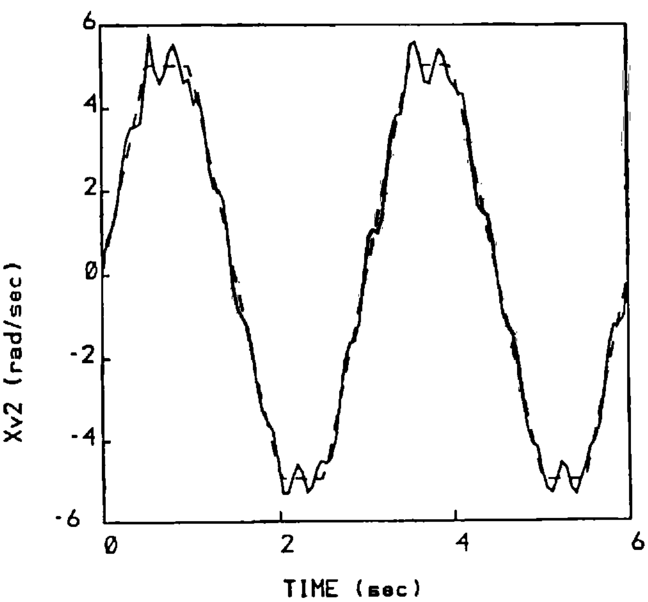



*Fig. 8. Velocity response of axis 2 for the DCAL (case* ii). *---desired trajectory; —actual trajectory.*

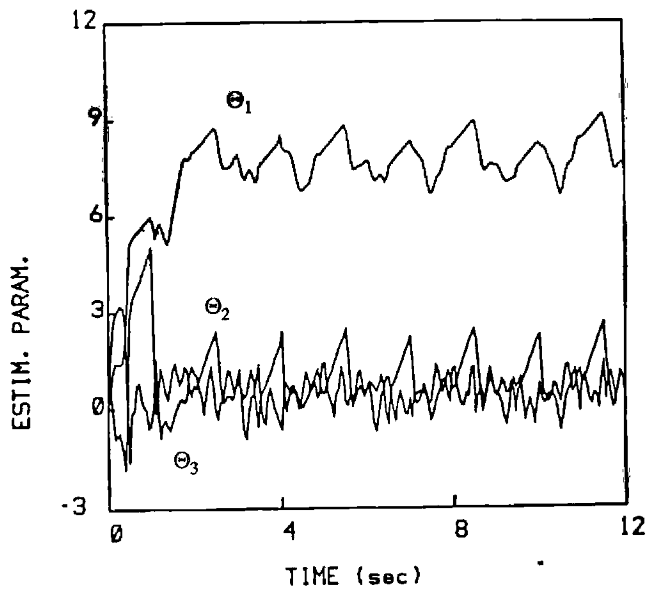



*Fig. 9. Parameter estimates for the ECAL (case* ii).

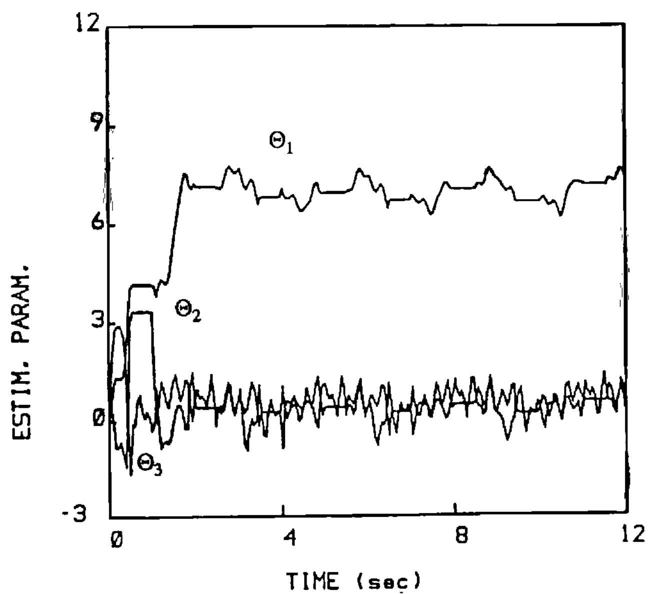



*Fig. 10. Parameter estimates for the DCAL (case* ii).

Simulation results revealed that the position and velocity response of the second axis contained a larger tracking error than those of the first axis. This is expected, since the same desired position and velocity trajectories were chosen for both axes, and the second axis has a smaller inertia than the first one. Hence the second axis is more sensitive to unmodeled disturbances and parameter variations. We will therefore only present the time response of the second axis and omit the corresponding results for the first axis.

The following control gains were used in the simulations: $\sigma_v = 40$, $\sigma_p = 20$, $\sigma_n = 10$, and $\lambda_p = 5$. Simula-

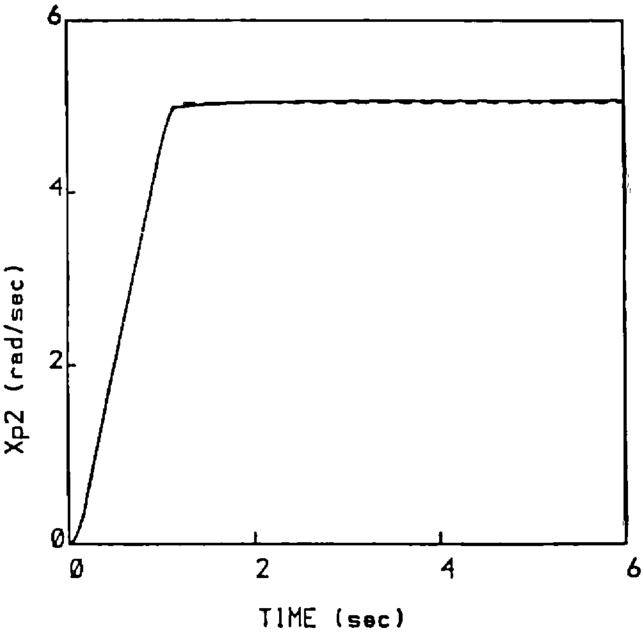



*Fig. 11. Position response of axis 2 for the ECAL (case iii). ---desired trajectory; —actual trajectory.*

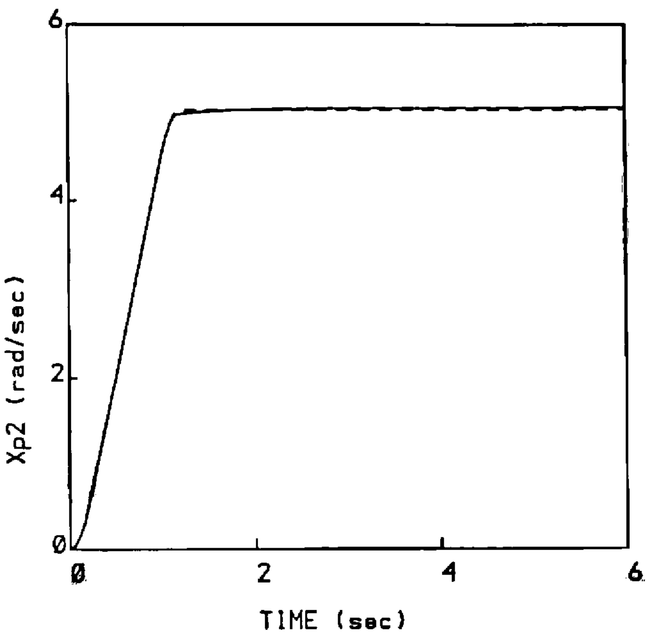



*Fig. 12. Position response of axis 2 for the DCAL (case iii). ---desired trajectory; —actual trajectory.*

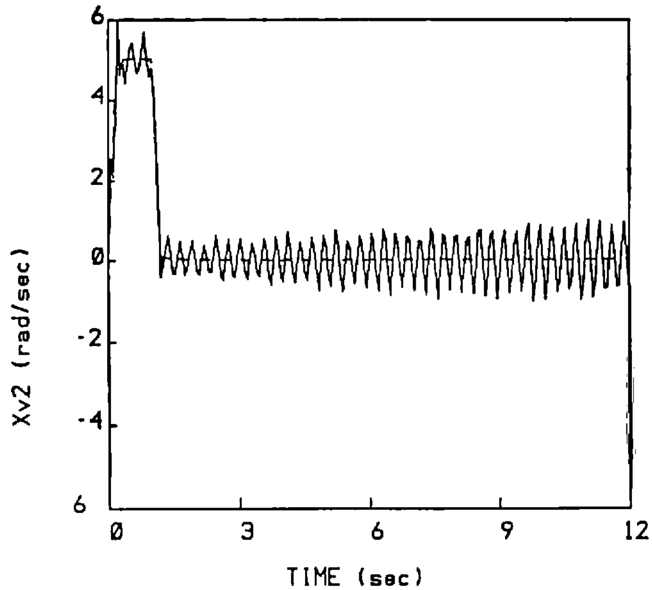



*Fig. 13. Velocity response of axis 2 for the ECAL (case iii). ---desired trajectory; —actual trajectory.*

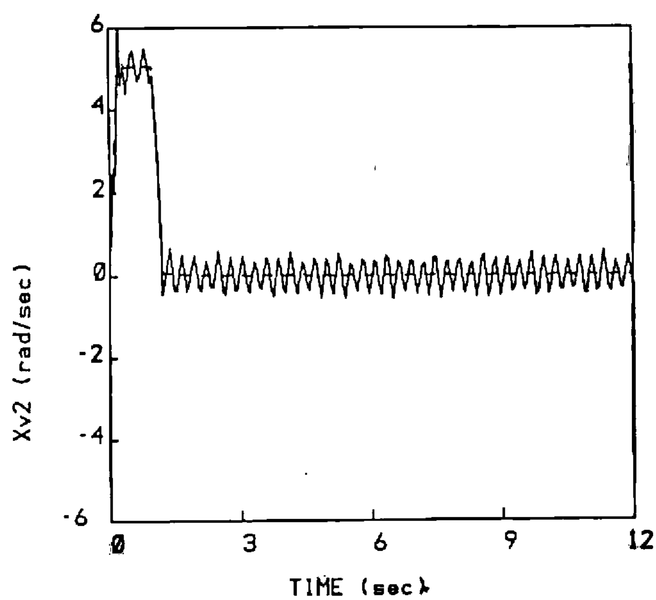



*Fig. 14. Velocity response of axis 2 for the DCAL (case iii). ---desired trajectory; —actual trajectory.*

tions were conducted for the following three cases:

Case *i*. Time varying load from 0 to 10 kg; no disturbances (Figs. 3-6).

Case *ii*. Time varying load from 0 to 10 kg with input and output sinusoidal disturbances. $|d_v| = 1$, $|d_p| = 0.5$, and $|d_q| = 5$ (Figs. 7-10).

Case *iii*. The same load and disturbances as in case *ii*, but with a desired position trajectory that converges to a constant (i.e., regulation problem) (Figs. 11-16).

Fig. 15. *Parameter estimates for the ECAL (case* iii).

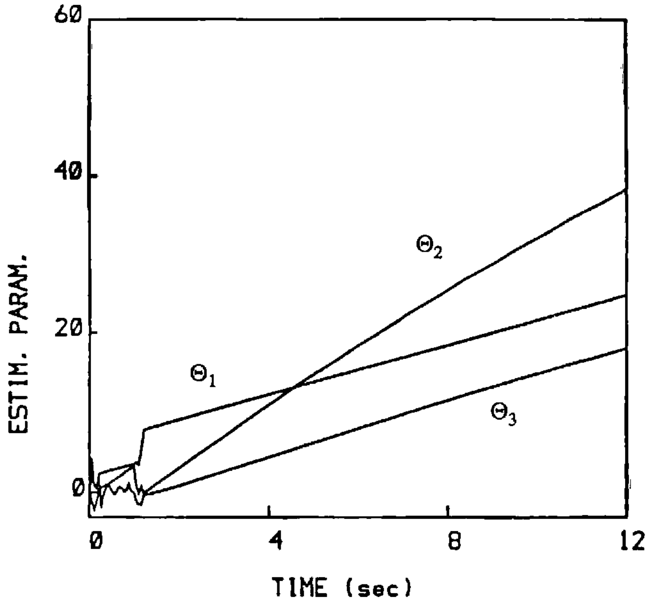
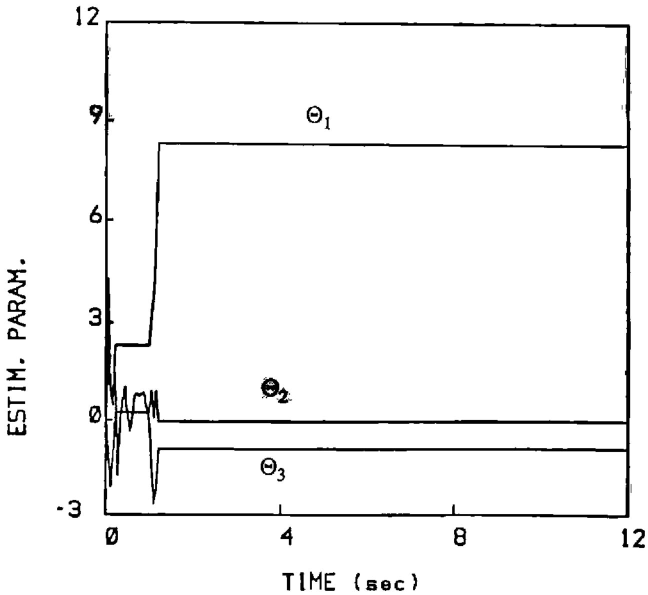



Fig. 16. *Parameter estimates for the DCAL (case* iii).




Simulation results show that the performance of the ECAL and the DCAL are identical in case *i*. Both position and velocity errors converge to zero, and the estimated parameters converge to their true values. With the addition of the disturbances in case *ii*, the velocity and position response of the ECAL and the DCAL still remain very close to the desired trajectories. However, the estimated parameters in the ECAL drift when the adaptation signal becomes constant. No drifts occurs in the DCAL. Case *iii* shows that the ECAL becomes unstable under the presence of noise and a constant adaptation signal. The DCAL maintains its stability and the parameters converge to constant values.

## 5. Conclusion

Additional stability and robustness properties of the adaptive scheme proposed by Sadegh and Horowitz (1987) were studied. Sufficient conditions for the convergence of the parameter error to zero were also derived.

A computationally efficient scheme, called in this article the desired compensation adaptation law (DCAL), was introduced. The DCAL utilizes the desired trajectory outputs instead of actual manipulator outputs in the parameter adaptation algorithm and the nonlinearity compensation controller. It was demonstrated that the DCAL inherits the stability properties of the original scheme, called in this article the exact compensation adaptation law (ECAL). Simulation studies revealed that the DCAL, in addition to being more computationally efficient, exhibits a better robustness with respect to disturbances, as compared with the ECAL.

## Appendix A

Definition 1: **Vector Norm**

For a vector $\mathbf{w} \in \mathbf{R}^n$, we define the norm of $\mathbf{w}$ to be the Euclidean norm, i.e.,

$$|\mathbf{w}| := \left[\sum_{i=1}^{i=n} w_i^2\right]^{1/2}. \tag{A.1}$$

Definition 2: **Matrix Norm**

For a matrix $\mathbf{M} \in \mathbf{R}^{m \times n}$ the norm is defined to be

$$|\mathbf{M}| := \sup_{|\mathbf{w}|=1} |\mathbf{M}\mathbf{w}|. \quad \text{(A.2)}$$

Definition 3: **Almost Everywhere Uniform Continuity (a.e.u.c)**

A function $\mathbf{f}(t) : \mathbf{R}^+ \to \mathbf{R}^n$ is said to be uniformly continuous almost everywhere *iff* for any given $t_0$ and any given $\epsilon$ there exist $\delta(\epsilon)$ such that:

$$|\mathbf{f}(t) - \mathbf{f}(t_0)| \leqslant \epsilon \quad \text{for all } t \text{ such that} \quad t \in [t_0, t_0 + \delta] \quad \text{or} \quad t \in [t_0 - \delta, t_0]$$

Definition 4: **Persistent Excitation**

A matrix function $\mathbf{W}(t) : \mathbf{R}^+ \to \mathbf{R}^{m \times n}$ ($m \leqslant n$) is said to be persistently exciting (P.E.) *iff* there exist a $\delta > 0$ and an $\alpha > 0$ such that for all $s \in \mathbf{R}^+$ we have:

$$\int_s^{s+\delta} \mathbf{W}^T(\tau)\mathbf{W}(\tau)d\tau \geqslant \alpha\, \mathbf{I} \quad \text{(A.3)}$$

Definition 5: $\mathbf{L}_p$ **Function Norm**

For a Lebesgue measurable function $\mathbf{f}(t) : \mathbf{R}^+ \to \mathbf{R}^n$, the $\mathbf{L}_p$ norm for $p \in [1, \infty)$ is defined to be:

$$\|\mathbf{f}(.)\|_p := \left[ \int_0^\infty |\mathbf{f}(\tau)|^p d\tau \right]^{1/p}. \quad \text{(A.4)}$$

For $p = \infty$ the norm is defined to be:

$$\|\mathbf{f}(.)\|_\infty := \sup_{\tau \geqslant 0} |\mathbf{f}(\tau)| \quad \text{almost everywhere.} \quad \text{(A.5)}$$

Definition 6: **Exponential Stability** [Bodson and Sastry (1984)]

Consider a system represented by the differential equation $\dot{\mathbf{x}} = f(\mathbf{x}, t, \mathbf{u})$.

*i.* $\mathbf{x} = 0$ is an exponentially stable equilibrium point of the unperturbed system (i.e., $\mathbf{u} = 0$) *iff* there exist $\gamma$ and $M > 0$ such that for all $t_0 \geqslant 0$ and $t \geqslant t_0$:

$$|\mathbf{x}(t)| \leqslant M\,|\mathbf{x}_0|\, e^{-\gamma(t-t_0)}, \; \mathbf{x}_0 = \mathbf{x}(t_0), \quad \text{(A.6)}$$

for any initial condition $\mathbf{x}_0$ in some closed ball of radius $\mathbf{h} > 0$ centered at 0.

*ii.* The system is said to be globally exponentially stable if eq. (A.6) holds for all $\mathbf{x}_0$ belonging to the set of allowable states.

Definition 7: $\mathbf{L}_p$ **Stability**

A dynamical system is said to be input/output $\mathbf{L}_p$ stable with respect to the pair $(\mathbf{u}, \mathbf{y})$ *iff* there are positive constants $\alpha_1$ and $\alpha_2$, such that:

$$\|\mathbf{y}(.)\|_p \leqslant \alpha_1 \|\mathbf{u}(.)\|_p + \alpha_2 \quad \text{(A.7)}$$

# Appendix B

Theorem 2, part *ii*

LEMMA

Let $f(t) : \mathbf{R}^+ \to \mathbf{R}^n$ be a uniformly continuous almost everywhere (u.c.a.e.) function; then for any $p_0 > 0$:

$$\lim_{t\to\infty} f(t) = 0 \quad \textit{iff} \quad \lim_{t\to\infty} \int_t^{t+p} f(\tau)d\tau = 0 \quad \text{for all } 0 < p \leqslant p_0.$$

PROOF

Suppose that $\lim_{t\to\infty} f(t) = 0$, then

$$\lim_{t\to\infty}\int_t^{t+p} f(\tau)d\tau = 0$$

is immediate. We will now show that

$$\lim_{t\to\infty}\int_t^{t+p} f(\tau)d\tau = 0$$

implies $\lim_{t\to\infty} f(t) = 0$ by contradiction.

Assume $\lim_{t\to\infty} f(t) \neq 0$; then there exists a $K$ such that, for any given $t_1$ there exists a $t_2 > t_1$ for which we have $|f(t_2)| > K$. This implies that at least one of the components of $f(t_2)$, say $|f^i(t_2)|$, $> K/\sqrt{n}$. Without loss of generality we may assume $f^i(t_2) > (K/\sqrt{n}) > 0$. Notice that the a.e. uniform continuity of $f(t)$ implies a.e. uniform continuity of $f^i(t)$. Therefore without loss of generality there exists a $\delta < p_0$ such that $f^i(\tau) > (K/2\sqrt{n})$ for all $\tau \in [t_2, t_2 + \delta]$. This implies:

$$\left|\int_t^{t+\delta} f(\tau)d\tau\right| > \left|\int_t^{t+\delta} f^i(\tau)d\tau\right| = \int_t^{t+\delta} f^i(\tau)d\tau > \frac{K}{2\sqrt{n}}\delta$$

But this contradicts the fact that $\lim_{t\to\infty}\int_t^{t+p} f(\tau)d\tau = 0$ for $p = \delta < p_0$, since $t_1$ is any arbitrarily large positive number. □

PROOF OF PART (A) OF THEOREM 2

Denote $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\tilde{\mathbf{\Theta}}(t)$ by $f(t)$. If we can show that $\lim_{t\to\infty}\int_t^{t+p} f(\tau)d\tau = 0$ for all positive $p \leqslant$ some $p_0$, then by the above lemma the proof is complete.

Rewriting eq. (20):

$$f_1(t) + f_2(t) = f(t), \tag{B.1}$$

where

$$\begin{aligned} f_1(t) &= \mathbf{M}(\mathbf{x}_p)\dot{\mathbf{e}}_v \\ f_2(t) &= \mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{e}_v) + \mathbf{F}_v\mathbf{e}_v + \mathbf{F}_p e. \end{aligned} \tag{B.2}$$

In part $i$ of the theorem we showed that $\mathbf{e}_v$ and $e$ converge to zero asymptotically. Thus $f_2(t)$ also converges to zero and by the lemma we have

$$\lim_{t\to\infty}\int_t^{t+p} f_2(\tau)d\tau = 0,$$

and utilizing integration by parts to integrate $f_1(t)$,

$$\begin{aligned}\int_t^{t+p} f_1(\tau)d\tau &= \mathbf{M}(t+p)\mathbf{e}_v(t+p) - \mathbf{M}(t)\mathbf{e}_v(t) \\ &\quad - \int_t^{t+p}\dot{\mathbf{M}}(\tau)\mathbf{e}_v(\tau)d\tau.\end{aligned} \tag{B.3}$$

Thus,

$$\begin{aligned}\left|\int_t^{t+p} f_1(\tau)d\tau\right| &\leqslant \lambda_M[|\mathbf{e}_v(t+p)| + |\mathbf{e}_v(t)|] \\ &\quad + p_0\left[\sup_{t\leqslant\tau\leqslant t+p_0}|\dot{\mathbf{M}}(\tau)|\,|\mathbf{e}_v(\tau)|\right].\end{aligned} \tag{B.4}$$

Notice that $\dot{\mathbf{M}}(\mathbf{x}_p)(t)$ can be written as:

$$\dot{\mathbf{M}}(t) = \sum_{i=1}^{i=n}\frac{\partial\mathbf{M}}{\partial x_{pi}}(t)x_{v_i}(t). \tag{B.5}$$

In part $i$ of the theorem we showed that $\mathbf{x}_v(t)$ is bounded, which implies the boundedness of $\dot{\mathbf{M}}(\mathbf{x}_p)(t)$. Therefore the right side of inequality (B.4) converges to zero, which means $\lim_{t\to\infty}\int_t^{t+p} f_2(\tau)d\tau = 0$. Thus, $\lim_{t\to\infty}\int_t^{t+p} f(\tau)d\tau = 0$, and by the above lemma the proof is complete. □

PROOF OF PART (B) OF THEOREM 2

First notice that $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\tilde{\mathbf{\Theta}}$ converges to zero, since the right side of

$$\begin{aligned}|\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\tilde{\mathbf{\Theta}}| &\leqslant |\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u}) - \mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)|\,|\tilde{\mathbf{\Theta}}| \\ &\quad + |\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\tilde{\mathbf{\Theta}}|,\end{aligned} \tag{B.6}$$

converges to zero.

Let $\mathbf{P}(s, t) = \int_s^t \mathbf{W}^T(\tau)\mathbf{W}(\tau)d\tau$ Since $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ is P.E., then for some $\delta > 0$ and all $t$, we have:

$$\mathbf{P}(s, s+\delta) \geqslant \alpha > 0.$$

From integration by parts and the parameter adaptation law, it can be seen that:

$$\begin{aligned}\tilde{\mathbf{\Theta}}^T(t)\mathbf{P}(s,t)\tilde{\mathbf{\Theta}}(t) &= 2\int_s^t \tilde{\mathbf{\Theta}}^T(\tau)\mathbf{P}(s,\tau)\mathbf{K}\mathbf{W}^T(\mathbf{x}_p, \mathbf{x}_v, \mathbf{u})\mathbf{e}_v(\tau)d\tau \\ &\quad + \int_s^t \tilde{\mathbf{\Theta}}^T(\tau)\mathbf{W}^T(\tau)\mathbf{W}(\tau)\tilde{\mathbf{\Theta}}(\tau)d\tau\end{aligned} \tag{B.7}$$

Letting $t = s + \delta$, the right side of the above equation converges to zero. Since $\mathbf{P}(s, s+\delta) \geqslant \alpha\mathbf{I} > 0$, then we conclude that $\tilde{\boldsymbol{\Theta}}(t)$ converges to zero asymptotically. □

PROOF OF LEMMA 1

In this lemma we show that by replacing $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)$ for $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})$, the resulting error $\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e})$ is bounded by the errors $\mathbf{e}_v$ and $\mathbf{e}$. What makes these bounds possible is the fact that both $\mathbf{M}(\mathbf{x}_p)$ and $\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \mathbf{x}_v)$ are $C^\infty$ functions of their variables. Moreover, the derivatives of these functions with respect to $\mathbf{x}_p$ are uniformly bounded. Therefore we can apply the mean value theorem (MVT) (Marsden 1974) to obtain the difference between $\mathbf{W}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d)\boldsymbol{\Theta}$ and $\mathbf{W}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v, \mathbf{u})\boldsymbol{\Theta}$. From eq. (53), it can be seen that

$$\begin{aligned}\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) = &[\mathbf{M}(\mathbf{x}_p)\mathbf{u} - \mathbf{M}(\mathbf{x}_d(t))\ddot{\mathbf{x}}_d] \\ &+[\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v) - \mathbf{v}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \dot{\mathbf{x}}_d)] \\ &+[\mathbf{g}(\mathbf{x}_p) - \mathbf{g}(\mathbf{x}_d)]\end{aligned} \tag{B.8}$$

Denoting each term inside the braces of eq. (A.1) by $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$, respectively, we obtain

$$\mathbf{t}_1 = [\mathbf{M}(\mathbf{x}_p) - \mathbf{M}(\mathbf{x}_d(t))]\ddot{\mathbf{x}}_d + \mathbf{M}(\mathbf{x}_p)(\mathbf{u} - \ddot{\mathbf{x}}_d) \tag{B.9}$$

Applying the MVT to $\mathbf{M}(\mathbf{x}_p)$ we obtain

$$\mathbf{t}_1 = -\left[\sum_{i=1}^{i=n} \int_0^1 \frac{\partial \mathbf{M}}{\partial x_{pi}}(\mathbf{x}_d + \xi\mathbf{e})d\xi e_i\right]\ddot{\mathbf{x}}_d + \lambda_p\mathbf{M}(\mathbf{x}_p)(\lambda_p\mathbf{e} - \mathbf{e}_v),$$

where the last expression was obtained from eq. (17). Now by taking the norm of eq. (B.9), it can be seen that:

$$-\mathbf{e}_v{}^T\mathbf{t}_1 \leq \lambda_p\mathbf{e}_v\mathbf{M}(\mathbf{x}_p)\mathbf{e}_v - \lambda_p{}^2\mathbf{e}_v\mathbf{M}(\mathbf{x}_p)\mathbf{e} + \mathrm{b}_{m1}(\ddot{\mathbf{x}}_d)|\mathbf{e}_v||\mathbf{e}|, \tag{B.10}$$

where

$$0 \leq \mathrm{b}_{m1}(\ddot{\mathbf{x}}_d) \leq \sup_{\mathbf{x}_p}\left[\sum_{i=1}^{i=n}\left|\frac{\partial \mathbf{M}}{\partial x_{pi}}\ddot{\mathbf{x}}_d\right|^2\right]^{1/2}. \tag{B.11}$$

Similarly the second term of (B.8), $\mathbf{t}_2$, can be expressed as follows

$$\begin{aligned}\mathbf{t}_2 &= [\mathbf{v}(\mathbf{x}_p, \dot{\mathbf{x}}_d, \dot{\mathbf{x}}_d) - \mathbf{v}(\mathbf{x}_d, \dot{\mathbf{x}}_d, \dot{\mathbf{x}}_d)] \\ &\quad+[\mathbf{v}(\mathbf{x}_p, \mathbf{x}_v, \hat{\mathbf{x}}_v) - \mathbf{v}(\mathbf{x}_p, \dot{\mathbf{x}}_d, \dot{\mathbf{x}}_d)] \\ &= \left[\int_0^1 \frac{\partial \mathbf{v}}{\partial \mathbf{x}_p}(\mathbf{x}_d + \xi\mathbf{e}, \dot{\mathbf{x}}_d, \dot{\mathbf{x}}_d)\mathrm{d}\xi\right]\mathbf{e} \\ &\quad+\mathbf{v}(\mathbf{x}_p, \dot{\mathbf{x}}_d, \mathbf{e}_v) - 2\mathbf{v}(\mathbf{x}_p, \dot{\mathbf{x}}_d, \lambda_p\mathbf{e}) - \mathbf{v}(\mathbf{x}_p, \mathbf{e}_v, \lambda_p\mathbf{e}) \\ &\quad+\mathbf{v}(\mathbf{x}_p, \lambda_p\mathbf{e}, \lambda_p\mathbf{e})\end{aligned} \tag{B.12}$$

Recalling that $v_i(\mathbf{x}_p, \mathbf{w}_1, \mathbf{w}_2) = \mathbf{w}_1{}^T\mathbf{N}^i(\mathbf{x}_p)\mathbf{w}_2$, then $\mathbf{e}_v{}^T\mathbf{t}_2$ can be expressed as

$$\begin{aligned}\mathbf{e}_v{}^T\mathbf{t}_2 \leq &[\mathrm{b}_{v1}(\dot{\mathbf{x}}_d) + 2\lambda_p\mathrm{b}_{v2}(\dot{\mathbf{x}}_d)]|\mathbf{e}_v||\mathbf{e}| \\ &+ \mathrm{b}_{v2}|\mathbf{e}_v|^2 + \lambda_p{}^2\mathrm{b}_{v3}|\mathbf{e}_v||\mathbf{e}|^2 \\ &+\lambda_p\mathrm{b}_{v3}|\mathbf{e}_v|^2|\mathbf{e}|,\end{aligned} \tag{B.13}$$

where

$$\begin{aligned}&0 \leq \mathrm{b}_{v1}(\dot{\mathbf{x}}_d) \leq \sup_{\mathbf{x}_p}\left[\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\left|\dot{\mathbf{x}}_d^T\frac{\partial \mathbf{N}^i(\mathbf{x}_p)}{\partial x_{pj}}\dot{\mathbf{x}}_d\right|^2\right]^{1/2} \\ &0 \leq \mathrm{b}_{v2}(\dot{\mathbf{x}}_d) \leq \sup_{\mathbf{x}_p}\left[\sum_{i=1}^{i=n}|\mathbf{N}^i(\mathbf{x}_p)\dot{\mathbf{x}}_d|^2\right]^{1/2} \\ &0 \leq \mathrm{b}_{v3} \leq \sup_{\mathbf{x}_p}\left[\sum_{i=1}^{i=n}|\mathbf{N}^i(\mathbf{x}_p)|^2\right]^{1/2}\end{aligned} \tag{B.14}$$

Finally, the third term, which is caused by gravity, can be rewritten using the MVT as

$$\mathbf{t}_3 = \left[\int_0^1 \frac{d\mathbf{g}}{d\mathbf{x}_p}(\mathbf{x}_d + \xi\mathbf{e})d\xi\right]\mathbf{e}. \tag{B.15}$$

After taking the norm, we obtain

$$\mathbf{e}_v{}^T\mathbf{t}_3 \leq \mathrm{b}_{g1}|\mathbf{e}_v||\mathbf{e}|, \tag{B.16}$$

where

$$0 \leq \mathrm{b}_{g1} \leq \sup_{\mathbf{x}_p}\left|\frac{d\mathbf{g}}{d\mathbf{x}_p}(\mathbf{x}_p)\right|. \tag{B.17}$$

By adding up the three terms obtained above, we get

$$\begin{aligned}-\mathbf{e}_v{}^T\Delta\mathbf{W}(\mathbf{e}_v, \mathbf{e}) \leq &\lambda_p\mathbf{e}_v{}^T\mathbf{M}(\mathbf{x}_p)\mathbf{e}_v - \lambda_p{}^2\mathbf{e}_v{}^T\mathbf{M}(\mathbf{x}_p)\mathbf{e} \\ &+\mathrm{b}_1(\dot{\mathbf{x}}_d)|\mathbf{e}_v|^2 + \mathrm{b}_2(\ddot{\mathbf{x}}_d, \dot{\mathbf{x}}_d)|\mathbf{e}_v||\mathbf{e}| \\ &+\mathrm{b}_3[|\mathbf{e}_v|^2|\mathbf{e}| + \lambda_p|\mathbf{e}_v||\mathbf{e}|^2],\end{aligned} \tag{B.18}$$

where $b_1$, $b_2$, and $b_3$ are bounded by the following quantities:

$$\begin{aligned} b_1(\dot{x}_d) &\le b_{v2}(\dot{x}_d) \\ b_2(\ddot{x}_d, \dot{x}_d) &\le b_{m1}(\ddot{x}_d) + b_{v1}(\dot{x}_d) + 2\lambda_p b_{v2}(\dot{x}_d) + b_{g1} \\ b_3 &\le \lambda_p b_{v3} \end{aligned} \tag{B.19}$$

□

## Appendix C

Derivation of Eqs. (2) and (3)

Consider a $n$ degree of freedom robotic manipulator composed by revolute or prismatic joints. The equations of motion for the arm are given by eq. (1). All the constraints in this system are holonomic and scleronomic and the joint positions vector $x_p$ is a generalized coordinate vector for this system (Rosemberg, 1977). Define the $n \times 1$ generalized force vector by $\tau(t)$. The kinetic energy of the system is given by

$$T = \frac{1}{2} x_v^T M(x_p) x_v. \tag{C.1}$$

Eq. (1) can be derived using Lagrange's equations of motion.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial x_v}\right) - \frac{\partial T}{\partial x_p} = \tau(t) \tag{C.2}$$

Expanding the first two terms of eq. (C.2) we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial x_v}\right) = M(x_p)\dot{x}_v(t) + \dot{M}(x_p)x_v(t) \tag{C.3}$$

$$\frac{\partial T}{\partial x_p} = \frac{1}{2}\left[x_v^T \frac{\partial M(x_p)}{\partial x_{p_1}} x_v \cdots x_v^T \frac{\partial M(x_p)}{\partial x_{p_n}} x_v\right]^T. \tag{C.4}$$

The last term in eq. (C.3) can expanded as follows

$$\begin{aligned} \dot{M}(x_p)x_v &= \left[\frac{\partial M(x_p)}{\partial x_{p_1}} x_v \cdots \frac{\partial M(x_p)}{\partial x_{p_n}} x_v\right] x_v \\ &= \frac{1}{2}\left[x_v^T\left[\frac{\partial m_1}{\partial x_p} + \left(\frac{\partial m_1}{\partial x_p}\right)^T\right] x_v \right. \\ &\quad \left. \cdots x_v^T\left[\frac{\partial m_n}{\partial x_p} + \left(\frac{\partial m_n}{\partial x_p}\right)^T\right] x_v\right]^T. \end{aligned} \tag{C.5}$$

Combining eqs. (C.2)-(C.5) we obtain the desired result:

$$M(x_p)\dot{x}_v(t) + v(x_p, x_v) = \tau(t), \tag{C.6}$$

where $v(x_p, x_v)$ is given by eqs. (2) and (3).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